Mathematics and Symmetry: A Bridge to Understanding

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Abstract

This paper describes how to teach group theory using symmetries of geometric shapes and artwork. Students travel on a road of discovery as they work on discovery projects to gain knowledge about abstract concepts.

1. Introduction

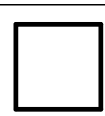
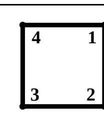
This article explores how projects using symmetry can be used in advanced mathematics classes as a means for students to learn abstract algebra. The projects are designed to implement my philosophy that the student should be an active partner in the discovery and learning process. Students use critical thinking skills to discover mathematical ideas. Traditionally, higher level mathematics is taught through lecture presentation. These projects provide an alternative approach by creating a student centered and hands-on classroom. The mathematics is the same; the flavor of the course is drastically different!

In order to provide an understanding of how these projects are constructed, the first one will be included in its entirety. The remaining projects will be illustrated by descriptions and brief excerpts.

2. Symmetries of a Square

Overall Description: The project starts with a simple square cut out of paper. Students label the vertices on both the front and back of the square and then use the symmetries of the square to create a set of elements and a binary operation on that set. The project enables students, in a straight forward manner, to construct an abstract group. When the theoretical definition of a group is formally introduced at the end of the activity, the students understand its meaning since they have experienced the construction of this mathematical object. As students work through the project, they communicate with each other to understand the ideas. These discussions provide a significant learning experience. When a group of students reaches a stumbling block, the instructor is available as a resource, responding to student queries with a series of leading questions that enable the students to answer their own questions. As you read through the project, notice that each vital concept is discussed using previously explored examples and then connecting the idea to the new setting. For example, students find additive and multiplicative identities for real numbers, a simple task, and then generalize the concept to determine the identity for the * operation on the symmetries of a square.

<u>Symmetries of a Square Project</u> This activity is designed to compel you think about mathematics using a system of objects other than numbers. We will move the square according to certain procedures we establish.

To begin cut out a square.		Next, label each corner of the square as indicated on the top picture. The back of the square also needs labels. Place 1' behind the 1, 2' behind the 2, 3' behind the 3, and 4' behind the 4.	

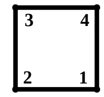
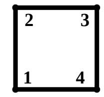
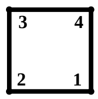
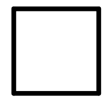
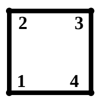
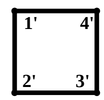
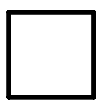
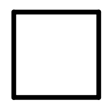
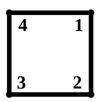
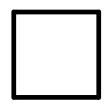
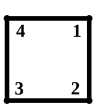
Let the standard position of the square have 1 in the top right hand corner. Place the square in front of you and put it in the standard position. We will now give names to several other positions. First, rotate the square 90^0 clockwise. You should obtain the position shown on the right. We will call this rotation of the square A.	
Place the square back into the standard position. Rotate the square 180^0 clockwise. We will call this rotation B. Use the descriptions provided in Chart I to complete the "resulting square" column.	

Chart I: The Symmetries of the Square

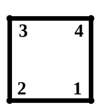
Name	Description	Resulting Square		Name	Description	Resulting Square
A	90^0 clockwise rotation			E	Rotate the square about a horizontal line through the middle of the square.	
B	180^0 clockwise rotation			F	Rotate the square about a vertical line through the middle of the square.	
C	270^0 clockwise rotation			G	Rotate the square about a line drawn from the upper left corner to the lower right corner of the square.	
D	360^0 clockwise rotation			H	Rotate the square about a line drawn from the lower left corner to the upper right corner of the square.	

We will now begin to construct a strange new mathematical system. Consider the set of elements {A, B, C, D, E, F, G, H} where each letter represents the symmetry of the square from the table you have constructed. We will define an operation on the elements of this set. The operation is denoted with the star symbol, *. B * A means A followed by B. Note that the * notation is a composition of functions (or transformations) so the element on the right goes first. Let's translate this into more understandable terms as illustrated in the next example. Please note that this operation is associative; however, we will not prove this property.

I. Place your square into standard position.



II. Apply A to the square from column **I** to obtain a new square.



III. Apply B to square from column **II**

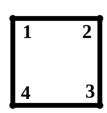


Chart I tells us the symmetry that is represented by the square in column III above is C. This means that B * A = C. We want to compute the * operation for all possible combinations of the elements in the set {A, B, C, D, E, F, G, H}. To fill in the * Operation Chart, start with the element in the first column and pair it with each element in the first row. For example, the spot for A * C is in the first row and third empty slot, but the spot for C * A is in the third row and first empty slot. **Complete the * Operation Chart." Be sure to check your answers with another student along the way.**

The * Operation Chart

*	A	B	C	D	E	F	G	H
A								
B								
C								
D								
E								
F								
G								
H								

Next we consider the interesting questions!

I. Closure: Determine if each of the following sets are closed under the indicated operation. If the set is not closed, provide an example to demonstrate why.

	Z **Integers**	$\mathbf{Z}^+$ **Positive Integers**	$\mathbf{Z}^-$ **Negative Integers**	**R** **Real Numbers**	**P** **Polynomials**	$\mathbf{P}_2$ **Polynomials of Degree** ≤ 2
Addition						
Subtraction						
Multiplication						
Division						

Give an example of a set consisting of only two numbers so that the set is closed under multiplication, but not closed under addition.

Determine if the set {A, B, C, D, E, F, G, H} closed under the * operation and explain your answer.

II. 1. Commutivity: In the set of real numbers, name two operations that are commutative. Give a specific example for each operation you name.

2. In the set of real numbers, name two operations that are not commutative. Give a specific example for each operation you name.

3. Use the chart to determine if the * operation is commutative on the set {A, B, C, D, E, F, G, H} and explain your answer.

III. Identity: Determine if each set contains an identity element for addition. Give a specific example to illustrate your answer.

	Z **Integers**	$\mathbf{Z}^+$ **Positive Integers**	$\mathbf{Z}^-$ **Negative Integers**	**R** **Real Numbers**	**P** **Polynomials**	$\mathbf{P}_2$ **Polynomials of Degree** ≤ 2
Identity for Addition						
Example						

Determine if each set contains an identity element for multiplication. Give a specific example to show your answer works. [Pick an example that none of your classmates will likely pick!]

	Z **Integers**	$\mathbf{Z}^{+}$ **Positive Integers**	$\mathbf{Z}^{-}$ **Negative Integers**	**R** **Real Numbers**	**P** **Polynomials**	$\mathbf{P}_2$ **Polynomials of Degree** ≤ 2
Identity for Multiplication						
Example						

What is the identity element for the set {A, B, C, D, E, F, G, H} and the * operation? Give four examples to illustrate that your answer is correct.

IV. Inverses What is the additive inverse of 7? What is the additive inverse of -5? Does every real number have an additive inverse? What is the multiplicative inverse of 7? __ What is the multiplicative inverse of -5? There is one real number that does not have a multiplicative inverse. What is the real number that does not have a multiplicative inverse? Why?

Complete the following chart as you explore inverses for the * operation.

Element	Does the element have an inverse?	List the inverse or explain why it does not have an inverse
A		
B		
C		
D		
E		
F		
G		
H		

A set of elements G and an associative operation * is called a group if

1. G has an identity element.
2. The composition of any two elements in G is also in G, and
3. For every element f in G, there is another element g in G so that f*g=g*f = 1.

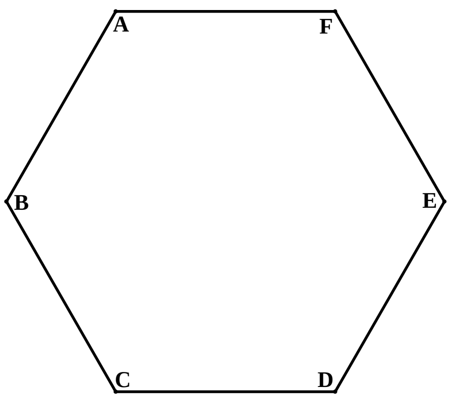
Is the set of symmetries of the square a group? Is $\mathbf{P}_2$ a group under the operation of addition? Is $\mathbf{P}_2$ a group under the operation of multiplication? Explain your answers.

3. Symmetries of Regular Polygons

This project not only provides the opportunity for students to explore the symmetries of regular polygons, but also to investigate the concept of a subgroup by utilizing a concrete example. For this activity, it is effective to divide the class into several groups, each group investigating the symmetries of a particular regular polygon, listing all of the rotations and reflections. Then each group shares their results with the entire class. A series of questions leads the students to discover that the particular subset of these symmetries consisting solely of the rotations forms a group on their own. The subset of the reflections does not share this property. Thus, the students discover a subgroup, once again using a hands-on example. The concept of commutivity is also explored as the students recognize that a subgroup can be commutative when the actual group itself does not have this property.

Here is the portion of the project for the hexagon group. Other groups explore a regular pentagon, heptagon, octagon, and nonagon.

Symmetries of a Regular Hexagon

<table>
<tr>
<td>
</td>
<td>Describe all rotational symmetries of the hexagon.
Hint: Place a dot in the center of the hexagon. Determine the number of degrees in each rotational symmetry.</td>
<td>Describe all reflection symmetries of the hexagon.
Hint: Draw all lines of symmetry.</td>
</tr>
</table>

A hexagon has __ sides, ____ rotational symmetries, and ____ reflection symmetries.

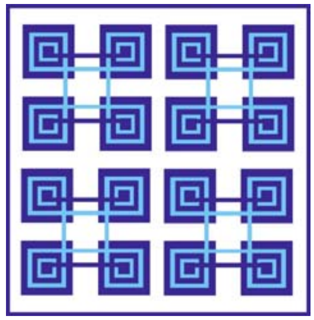
Is the set of symmetries of the hexagon a group? A commutative group? Is the set of rotational symmetries of the hexagon a group? A commutative group? Is the set of reflection symmetries of the hexagon a group? A commutative group? Explain your answers.

After each group of students shares their results on the regular polygon their group has explored, we create a summary chart for the results. The final row asks the students to generalize the results for a regular n-gon.

Regular Polygon	Number of Rotational Symmetries	Number of Reflection Symmetries	Total Number of Symmetries

4. Symmetries of Artwork

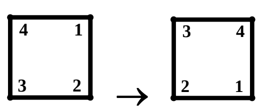
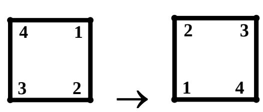
The purpose of this project is two-fold. First, the students use mathematical analysis to examine artwork. Second, the students expand their appreciation of mathematics as part of the world around us, even when it might be less than obvious. This project is strikingly different from the previous projects where the goal was to learn and experience abstract mathematical ideas. Here the activity provides the student with the opportunity to explore the world of art from a mathematical perspective, creating a bridge between art and mathematics. Students are required to find three different artists and prepare a PowerPoint presentation of approximately 10 minutes on a particular piece of work from each artist, describing the symmetry group for that piece of artwork. Consider the following example from the paper of one graduate student. The design is by Ismail Rashada.



"The design was created digitally by using Adobe Illustrator. Each square unit is composed of five nested squares. . . .This relationship of blocks can be found in Islamic art in architectural monuments." The student then proceeds to explain the symmetries of the design.

5. Matrix Representations of D_4

This project uses the now very familiar symmetries of the square group to explore the connection between 2 x 2 matrices and symmetries, providing a natural example of an isomorphism. Students begin by assuming the origin is the center of the square and each side is 2 units in length. Students then label the coordinates of each vertex. By expressing each symmetry in cyclic notation, students relate the symmetry to a particular permutation, thus creating a relation between the symmetry group of the square, $\mathbf{D_4}$, and a subgroup of the general permutation group, $\mathbf{S_4}$. Next, students find the image of each vertex under the given symmetry and use this to find the matrix representation of the symmetry. The matrix can be checked for accuracy by using it to determine the image of each vertex. This activity enables the students to readily visualize the connection between certain groups of 2 x 2 matrices and $\mathbf{D_4}$. Students also verify the relationships between the matrix representations. For example, the square of each reflection is the identity transformation. The square of the matrix representing the reflection is also the identity matrix. Since a 180^0 rotation is a 90^0 rotation repeated two times, multiplying the matrix representing the 90^0 rotation by itself should give the matrix representing the 180^0 rotation. Although this type of verification is readily done in a traditional lecture mode, the students gain a far clearer and deeper understanding when they must rely on their own work to master the ideas.

Description	Image of Square	Image of (x,y)	Cyclic Notation	Matrix Notation
90^0. counterclockwise		(x, y)→	(1234)	
180^0 counterclockwise		(x, y)→	(13)(24)	

6. Putting Them All Together

Recall the following definitions:

- $\mathbf{S_n}$ represents the group of permutations of the set {1, 2, 3, . . .n}
- $\mathbf{D_n}$ represents the group of symmetries of a regular n-gon, the dihedral group
- $\mathbf{R_n}$ represents the group of rotations of a regular n-gon
- $\mathbf{A_n}$ represents the group of all even permutations in S_n, the alternating group

This project has the students use the very familiar symmetries of the square to visualize and understand S_4, D_4, R_4, and A_4. The students use this example to gain a better understanding of the more general groups S_n, D_n, R_n, and A_n.

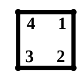
I. Fill in the blanks.

S_n is the set of permutations of {1, 2, 3, . . . n}	D_n is the set of symmetries of a regular ________
R_n is the set of ______ of a regular _____	A_n is the set of __ permutations of {1, 2, 3, . . n}

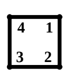
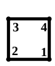
Using the definitions of S_n, D_n, R_n, and A_n, to fill in the following blanks.

____ ⊆ ____ ⊆ ____	____ ⊆ ____

II. Let's look closely at these relationships for n = 4. Consider a square with each vertex representing an integer from 1 to 4 in the representation shown.



Fill in the following table for S_4. Two examples have been completed for you.

Description as Product of Transpositions	Picture of Permutation	List the groups that contain this permutation. D_4, A_4, and R_4 If the permutation is in D_4, describe the symmetry it represents.	Description as Product of Transpositions	Picture of Permutation	List the groups that contain this permutation. D_4, A_4, and R_4. If the permutation is in D_4 describe the symmetry it represents.
(12)(12)		D_4, A_4, and R_4 identity	(132)		
(34)			(1342)		
(23)			(13)		
(234) = (24)(23)			(134)		
(243)			(13)(24)		
(24)			(1324)		
(12)			(1432)		
(12)(34)			(142)		
(123)			(143)		
(1234) =(14)(13)(12)		R_4, D_4 ,rotation of 90^0 clockwise	(14)		
(1243)			(1423)		
(124)			(14)(23)		

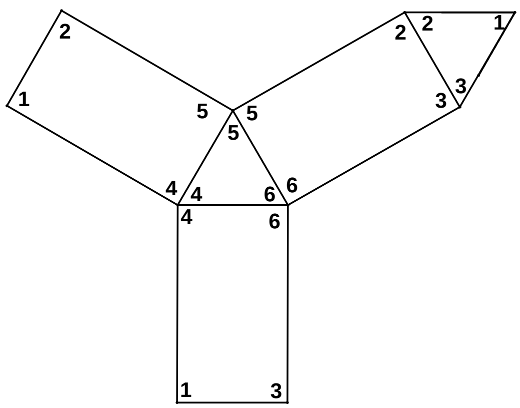
III. Complete the table.

Group	Order of Group
S_4	
D_4	
A_4	
R_4	

Describe the symmetries of the square that correspond to A_4.
How many odd permutations are there in S_4?
How many even permutations are there in S_4?

7. Symmetries of Three Dimensional Objects

The next project uses a regular triangular prism. Each student is provided a net from which to construct the object.



Through a series of questions, the students recognize that since there are six vertices, every symmetry can be represented as an element of S_6. However, it is not possible for every permutation in S_6 to be represented as a symmetry of the prism. A permutation is <u>physically realizable</u> if it can be demonstrated with the model. Once again, the students use an actual physical object and the symmetries of that object to expand their theoretical understanding.

8. Conclusions

Although manipulatives are quite common in elementary school mathematics, they are used far less frequently at the secondary and post-secondary levels. Using a hands-on approach deepens the level of understanding of our students, helping to make abstract concepts seem more concrete. Perhaps this is best illustrated by the following comment from a middle school mathematics teacher taking this class. ". . . each concept came alive through the use of manipulatives, games, and projects. . . [The instructor]. . . was able to relate every abstract concept to a concrete model so that every student understood what and why the topic was important in our teaching." The student not only learns the mathematics and its relationship to symmetry, but also takes pleasure as an active participant in the discovery process.