Building Models to Transition From Dimension to Dimension

Robert McDermott
Center for High Performance Computing
University of Utah
Salt Lake City, Utah, 84112, USA
E-mail: mcdermott@chpc.utah.edu

Abstract

Physical models are valuable for conveying spatial concepts in geometry. In this paper, I discuss how to build 4 models which transition between spatial dimensions 0 to 1, 1 to 2, 2 to 3, and 3 to 4. Supplies for these models are bungee cord and PCV pipe and connectors. Another supply is a hollow plastic sphere made from two hemispheres. These supplies cost me less than $60. The tools used are a measuring tape, a jig saw, a circular saw, a sander, a drill and bits, a rasp and scissors. A teacher who is proficient with power tools is capable of safely producing the pieces for these models. Students under a teacher's supervision are fully capable of assembling and moving these models. Students work together to expand their understanding of space by assembling and moving these models.

1. Introduction

Physical models for use in K-12 classrooms can be built from paper to prove an existence of 5 and only 5 regular solids [1], or from sticks to exhibit many symmetry relationships for both Platonic and Archimedean solids [2]. Even though these models are static, they convey considerable knowledge to a wide range of students who visually study them and physically manipulate them. These models are meaningful because they are based on very clear underlying foundational geometric concepts. The models described in this paper go beyond static to provide a dynamic experience for students. In addition, these dynamic models provide the opportunity for more than one student to cooperate and work together when moving the models. Each of the models bridges the gap from one spatial dimension to another.

Models employing these transitional concepts were first built in the summer of 1991 at an NSF supported Regional Geometry Institute held in Park City, Utah. My investigation into building these dimensional transition models was instigated by Herb Clemens [3], who was the director of the Institute and a faculty in the Department of Mathematics at the University of Utah at that time. Firstly, the goal for the models was to have attendees carry geometric models in a 4th of July Parade down Main Street of Park City. Secondly, there was interest in the K-12 teachers being able to build these models when they returned to their classrooms. Giving consideration to the capabilities of K-12 students, it was decided that the teachers would cut all the pieces. Subsequently, these pieces could be assembled by K-12 students in their classrooms.

The supplies of bungee cord, a plastic sphere, and PVC pipe with connectors are readily available at hardware stores or plumbing supply stores at a cost of less than $60. The tools used are a measuring tape for determining the lengths of cords and pipes, a jig saw to cut the pipe and elbows, a circular saw to cut a lengthwise slot in the pipe, a sander to clean printing off the pipe and connectors, a rasp to smooth the saw cuts, vice grip pliers to hold one end of the cord while threading and knotting, and scissors to cut the cord. These tools exist in many home tool boxes or can be rented at a reasonable cost from a local rental store. Teachers who are accustomed to fixing things around their homes may be experienced enough with using power tools, as well as being safety conscious. This fix-it experience prepares a teacher to prepare all the pieces for these 4 models. Students under a teacher's supervision are fully capable of assembling and moving these models. As many as 16 students can easily move the 4th model through its transition.

It is important to think of these physical models as "sketches" of the concept of transitioning between dimensions. The perfection associated with geometric concepts cannot be achieved with physical models. A failure to achieve perfection is something to get used to when building physical models with moderately-priced supplies and tools. As a consequence, the less exact the pieces, the more of a "sketch" the model becomes. However, these "sketches" fully support concepts and are useful examples for discussing spatial relationships. The models are all derived using the very basic elements of space. There is a ***point*** or small sphere for dimension 0, a ***line*** or an ***edge*** for dimension 1, a ***plane*** or a ***square*** for dimension 2, and a ***cell*** or ***cube*** for dimension 3. Lastly, this set of 4 models concludes with a ***hypercube*** for dimension 4. Two intersecting ***cubes*** have been chosen to represent a ***hypercube***. Each model can be moved to represent a dynamic transition between a pair of dimensions; 0 to 1, 1 to 2, 2 to 3, and 3 to 4.

The better a student understands these models and their dynamic properties, the better that student is able to understand and imagine more sophisticated concepts about space. Additional understanding of these models is achieved by the addition of touch and movement. When one or more students handle these models, they reinforce the knowledge gained through their sight. The size of the models is also important. It has been studied by J.J. Gibson, [4] that a model which is manipulated in the hands provides insight. These models are built to be handled by more than one student at a time to move them through their transitions. A model of modest size provides multiple students the opportunity to easily transition them from 1 dimension to the next, providing students many views. Students who are not participating with moving a model get the chance to observe the over-all shapes produced when moving the model. Taking turns with different students moving the models and other students observing is instructive. A geometric relationship that exists in many places in a model can be seen nearly simultaneously when it is in motion. This multiplicity of views reinforces the multiplicity of relationships in the models.

All edges in these models are the same length. All cords are cut to the same length and tied with a square knot. Once there is an understanding of using the cord to connect two edges, all connections with cord are the same. Initially when elements of the models are at rest, the elements are adjacent to each other. As the elements are separated, they move apart from each other by the length of an edge. All models are assembled and transitioned in a very similar manner. Pairs of edges are combined with each other to form a pair of ***lines***, a pair of ***squares***, and a pair of ***cubes*** to transition to a ***square***, a ***cube*** and a ***hypercube***.

The remainder of this paper provides descriptions for constructing cord and pipe models that will exhibit transitions between dimensions in space. These models serve to highlight foundational elements in space and a dynamic transition from dimension to dimension. A table itemizes the supplies for all models in Appendix I. This table of supplies has both the length of cords and pipes and the quantities of cord, pipes and connectors making it easier to be convinced that building the models is a reasonable and manageable task to accomplish.

2. Supplies and Tools

Supplies for these models consist of thin bungee cord (7/64") and PVC pipe and elbows. The PVC pipe was chosen at a small diameter ½" to be matched with the 22" length of the edges which produces reasonably durable models for teachers and students. A 3" diameter sphere which divides into hemispheres was chosen to be handled and easily observed by a class of 20 students. PVC connectors are used to allow the bungee cord to slide easily out of the pipe when ***edges*** separate, and back into the pipe when ***edges*** move back together. When the model is at rest the bungee cord is entirely concealed, allowing the students to concentrate their full attention on the basic elements of space; a ***point***, an ***edge,*** a ***square***, a ***cube*** or a ***hypercube***.

A tape measure is used to measure the length of each cord and piece of pipe. With a jig saw each piece of pipe is cut to length with clean cuts. The jig saw is also used to cut elbows into three pieces of 2 sleeves and a corner. Cuts are made straight across the pipes in order to form 90° right angles when joined together with the connectors. A rasp is used to smooth the ends of pipes before inserting them into connectors. Slots are cut lengthwise along pipes with a circular saw wide enough to accommodate the bungee cord. A sander cleans the printing from pipes and connectors and reduces the outside diameter of pipes near their ends, easing their entry into the connectors. Finally, a power drill with a ¼" bit is used to drill holes in connectors through which the bungee cord is threaded.

3. Models for the Transition Between Dimensions 0, 1, 2, 3, and 4

3.1. A Model to Transition Between Dimension 0 and Dimension 1. This model represents a transition from a ***point*** to a ***line*** and is built from 4 pieces. A first model piece is a hollow rigid plastic ball consisting of 2 hemispheres which can be separated from each other. This piece represents a ***point*** as the basic element of 0 dimensional space. Each plastic hemisphere has a hole drilled in it, **Figure 1a.** A second piece of the model is a PVC elbow that is sawed into 3 pieces, 2 sleeves and a corner, **Figure 1b**. A sleeve is glued inside the hemisphere, **Figure 1c.** A 12" length of cord is threaded through the holes in the plastic ball and sleeves then knotted with an over hand knot. It is long enough to stretch out to 22" when the two halves of the hemispheres are fully separated **Figure, 1d**. The sphere needs to be large enough to accommodate the cord when the 2 hemispheres are together. It also needs to be durable enough to be handled by K-12 students. The sleeves are shaped with the rasp to each fit inside the plastic hemispheres. A fourth piece of the model is a 22" length of pipe with straight end cuts, smoothed with a rasp and a slot cut along its length. This slot allows the pipe to be slipped over the cord when the hemispheres are fully separated and easily removed from the cord allowing the hemispheres to come together for a sphere. The slotted pipe provides a rigid edge to hold the 2 hemispheres apart at a distance that will be common for all edges in this set of transitional models. This ***edge,*** of course, represents a ***line,*** the basic element in 1 dimensional space.

In order to prepare the plastic sphere to become part of the model, each of the hemispheres has a hole drilled in it, **Figure 1a.** An elbow sleeve is glued to the inside of a hemisphere, **Figure 1b.** The 12" piece of cord is threaded through a sleeve and a hole and tied with an overhand knot, **Figure 1c.**

(a) **(b)** **(c)** **(d)**

Figure 1: 1 Hole in ball **(a)**, elbow & sleeves **(b),** sleeve glued in ball **(c),** cord and ball **(d).**

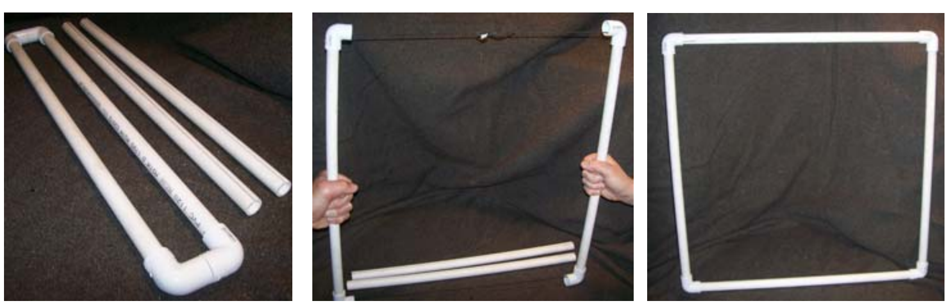
The dynamic aspect of this model illustrates the transition from a 0 dimensional ***point*** to a 1 dimensional ***line.*** Initially, two hemispheres are together forming a sphere or a ***point*** will be combined with a slotted pipe for an ***edge,*** **Figure 2a**. As the two hemispheres are separated the cord appears, stretching to a length of 22”. It takes two students to open and separate the two hemispheres and stretch the cord. At this time a third student is needed to slip the slotted pipe over the cord and inside the sleeves that were glued inside the hemispheres, **Figure 2b**. The pipe is able to withstand the compression created by the cord, producing a stable model, **Figure 2c**. This stability allows the model to be observed in the static form of a ***line***. The 2 hemispheres can be easily separated, the slotted pipe removed, and the hemispheres brought together again, reforming a sphere or a ***point***.

(a) **(b)** **(c)**

Figure 2: 1 A ***point*** and an ***edge*** **(a)**, stretching the bungee cord **(b),** a static ***line*** model **(c).**

3.2. A Model to Transition Between 1 Dimensional Space and 2 Dimensional Space. This model, representing the transition from a ***line*** to a ***plane,*** is built from 9 pieces: a 47” length of cord, 2 22” lengths of pipe, 4 90° elbows, and 2 22” lengths of slotted pipes. Pipe is sawed into 4 22” lengths with their ends smoothed. Two of the 4 ***edges*** will have the 4 elbows placed on their ends. With the elbows in place, the 47” length of cord is threaded through the elbows and pipes and tied with a square knot. Since the cord is shorter than twice the length of the pipes, plus the length of the elbows, a vise grip pliers is used to hold one end while the threading and some stretching occurs and the ends of the cord are tied together in a square knot. After the square knot is tied, the knot of the cord is slid inside one piece of pipe near its mid-point. These two pieces of pipe represent 2 ***edges*** for the opposite sides of a ***square***.

When this model is at rest 2 ***edges*** are side by side, **Figure 3a.** For the dynamics of this model, 4 students are most helpful. Two students separate the two ***edges,*** positioning them parallel to each other by stretching the cord by the length of the slotted pipe, **Figure 3b**. With these two ***edges*** separated, another 2 students position the 2 slotted ***edges*** over the cord and slide their ends into the elbows**.** With the slotted ***edges*** in place a stable ***square*** is formed, **Figure 3c**. This ***square*** represents a ***plane,*** a basic element of 2 dimensional space. Taking the static form of the model, students slightly separate the initial ***edges*** providing room to remove the slotted ***edges***, and once again returning them to their initial relationship.



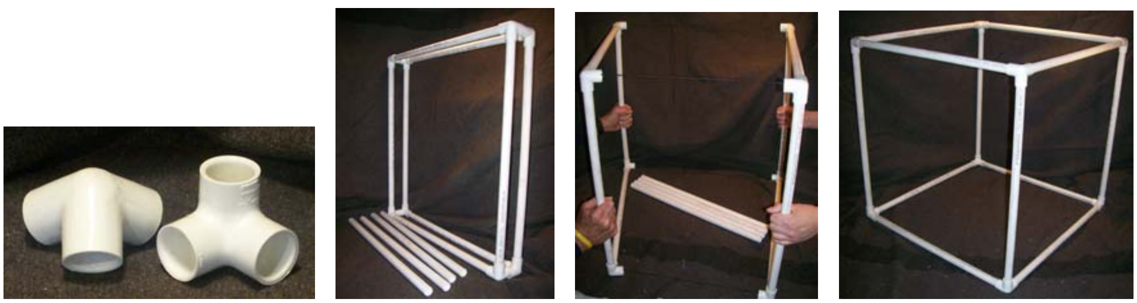
(a) **(b)** **(c)**

Figure 3: 2 ***edges*** with a loop of bungee cord (**a),** 2 ***edge*** as sides of a square **(b)**, 4 ***edges*** of square **(c)**

3.3. A Model to Transition Between 2 Dimensional Space and 3 Dimensional Space. This model representing the transition from a ***plane*** to a ***cell,*** is built from 22 pieces: 2 47” lengths of cord, 8 22” lengths of pipe, 8 3-way connectors, and 4 22” lengths of slotted pipe.

Pipe is cut and smoothed into 12 22” lengths, and 2 ***squares*** are assembled from 8 pieces of pipe and 8 3-way connectors, **Figure 4a.** The assembled ***squares*** are positioned along side each other for threading and knotting with 2 47” pieces of cord. The 2 square knots are slid inside pipes. The pairs of ***edges*** corded together serve as ***edges*** for opposite ***faces*** of a ***cube*****.** This ***cube*** represents a ***cell,*** the basic element in 3 dimensional space.

When this model is at rest 4 pairs of ***edges*** are are side by side, as the 2 ***squares*** are along side of one another Figure **4b.** For the dynamics of this model, 4 students separate the 2 ***squares*** so that they are parallel to each other and positioned to be the opposite faces of a ***cube,*** **Figure 4c**. Now that the initial two ***squares*** have been separated, another four students position the 4 slotted ***edges*** with ends being placed inside the open hole of the 3-way connector, **Figure 4d.** With the 8 slotted ***edges*** in place a stable ***cube*** is formed. Taking the static model, the 4 students can slightly separate the initial ***squares***, providing room for the slotted ***edges*** to be removed, allowing the initial ***squares*** to come back together, and returning the ***squares*** to a resting model.



(a) (b) (c) (d)

Figure 4: 2 3-way connectors **(a),** 2 ***squares*** **(b),** 2 ***squares*** as sides of a ***cube*** **(c)**, 12 ***edges*** of ***cube*** **(d).**

3.4. Transition Between 3 Dimensional Space and 4 Dimensional Space. This final model representing the transition from a ***cell*** to a ***hypercell,*** is built with 52 pieces: 4 47” lengths of cords, 24 22” lengths of pipe, 16 3-way connectors, and 8 22” lengths of slotted pipe. Pipe is prepared for 32 22” lengths to build 2 ***cubes*** and for the ***edge*** interconnections between the 2 ***cubes***. Slots are made in 8 22” pipes with circular saw. Holes are drilled in the inside of 8 3-way connectors, **Figure 5a,** and in the outside of 8 3-way connectors, **Figure 5b**.

(a) (b) (c)

Figure 5: Inside 3-way **(a),** outside 3-way **(b)**, cube-2 3-way inside cube-1 3-way **(c)**

To build a ***cube*** 12 pieces of pipe, 4 inside hole 3-way connectors and 4 outside hole 3-way connectors are used, **Figure 6a.** The lower left corner of **cube-1** is a 3-way connector with an **i**nside hole. In the square at the lower level of **cube-1** there are 2 more 3-way connectors with an **i**nside hole, and 1 3-way connector with an **o**utside hole. In the square at the upper level of **cube-1** there are 3 **i**nside and 1 **o**utside 3-way connectors. 4 pipes are inserted in 3-way connectors to separate the 2 ***squares*** for a ***cube***. Building a 2nd ***cube*** may require using a schematic diagram, **Figure 6b,** for the locations of the 8 3-way connectors for the **i**nside holes and the **o**utside holes. **Figure 6c** provides the positioning between the 2 ***cubes.*** To start building ***cube-2*** the 1st 3-way corner is positioned inside the volume of ***cube-1*** **Figure 5c.** This first corner is an **o**utside holed 3-way. The 1st ***edge*** extends to the right and lies on top of an ***edge*** in the 1st ***cube-1.*** A 2nd ***edge*** extends forward and also lies on top of an ***edge*** in the 1st ***cube-1.*** The third ***edge*** of this 1st 3-way connector extends upward and the top of this ***edge*** lies outside the volume of ***cube-1.*** The remaining 7 3-way connectors of ***cube-2*** are outside the volume of ***cube-1.***

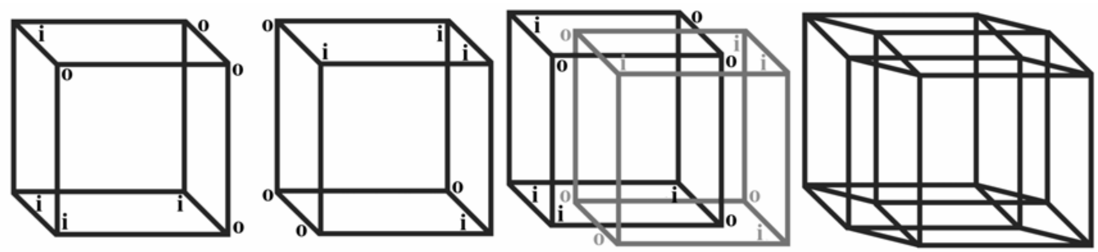


Figure 6: **cube-1 (a), cube-2 (b), cube-1 and cube-2 (c) hypercube (d).**

Two ***cubes*** are assembled so that the lower left corner of the ***cube-2*** lies close to and inside ***cube-1.*** An upper right corner of ***cube-1*** is inside ***cube-2,*** so that the cubes are interlocked and cannot be separated. Notice that the holes in the 3-way connectors of ***cube-1*** are matched up with the holes in the 3-way connectors of ***cube-2,*** that is, an inside hole is paired with an outside hole. This matching of holes will accommodate the cord for threading the ***vertical edges.*** The result is that 4 pairs of ***vertical edges*** will be corded together, **Figure 6d and Figure 7a**. To separate the 2 ***cubes*** 8 students each grasp a ***vertical edge*** and move apart for each other**.** Another 8 students place 8 slotted ***edges*** over cords to hold the 2 ***cubes*** apart**.** When separating the 2 ***cubes*** it is helpful to have 2 shoe boxes to prop ***cube-2*** up off the floor, **Figure 7b**. These 2 separated and propped up ***cubes*** with slotted ***edges*** over cords represent a ***hypercube*.** This ***hypercube*** model is another stable static model.

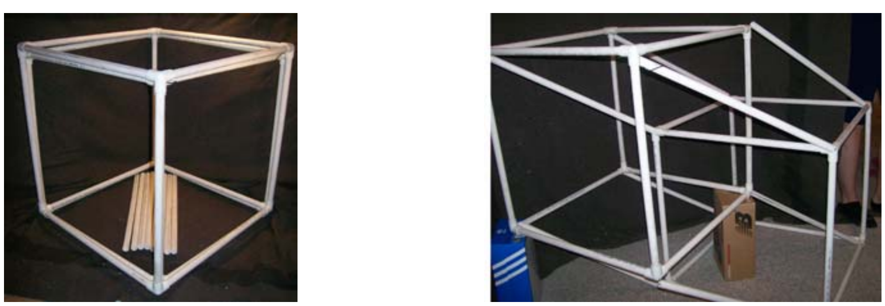


Figure 7: 2 ***cubes*** **(a),** 2 ***cubes*** separated with 8 slotted ***edges*** for a ***hypercube*** **(b).**

This ***hypercube*** model can have the 2 ***cubes*** slightly separated allowing the 8 slotted edges to be removed. The students holding corded ***edges*** can move toward each other until the ***edges*** are side by side. The 2 ***cubes*** are once again positioned with 8 pairs of ***edges*** next to each other, **Figure 7a,** as in the resting model.

4. Handling Pipe Models

I particularly like this set of 4 models because they use so many students in their transitioning. More students are used for positioning and removing slotted edges. The larger the model, the more students become helpful. When these models were originally built with a 4' edge of pipe and cord for the parade, there was one person to handle each corded ***edge***. The slotted edges were not part of the models for the parade because the handlers were in constant motion walking down the street, and the models were in constant motion transitioning. Everyone along the parade route saw something different. I believe that 1 student per slotted edge is needed to make subtle adjustments while the slotted edges are positioned over the cord and inserted into the sleeves. This would mean that a total of 16 students transition the ***hypercube***. Building multiples of the simpler models would increase the number of students participating.

5. Conclusion

These models were first built more than 15 years ago with the assistance of K-12 teachers. The models were well received by these teachers, the graduate students and research faculty working in geometry at the Institute. I will take this current set of models to the McGillis School in Salt Lake City, Utah for a presentation for middle school students and their teacher. Some models will be pre-assembled in their resting state and some models will be taken as pieces to be assembled completely by the students.

I have given different presentations with models for more than 10 years at this school. These presentations have been invaluable to the students who both listen well and anticipate the opportunity to handle models. In addition, they also help me learn more about the effectiveness of particular models in the classroom. The tactile experience of handling the models provides the students with a more engaging experience than merely viewing three-dimensional images on paper or on a computer monitor.

Models and presentations in classrooms over more than 20 years have proven to not only to be well received, but they are valuable experiences for me as well. When I am building models I am imagining them being handled by students and presenting the associated concepts in a classroom for students and teachers. It takes time and effort to build models and the hope is always that they will be effective in helping to clearly convey underlying geometric concepts. However, with each new set of models the testing ground is still students in the classroom and the models are only considered successful if the presentations and models are both thought provoking and engaging for students and teachers.

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References

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Appendix I

4 Models	**pipe**	**length**	**slotted**	**length**	**knot**	**elbow**	**3-way**	**3-way**	**Model in Lower**	**Model in Higher**
dimension	**count**	**inches**	**pipe**	**inches**	**tied**				**Dimension**	**Dimension**
0 to 1			1	22	2	1			2 hemispheres	edge
1 to 2	2	22	2	22	1	4			2 edges	square
2 to 3	8	22	4	22	2		8		2 squares	cube
3 to 4	24	22	8	22	4			16	2 cubes	hypercube
						buy 5		**buy 24**		
subtotal	34	748	15	330		**elbows**		**3-ways**		
total pipe	49	1078	~90’	**buy**	**5@20’**	**or**	**100’**	**pipe**		
cord	1@24”	7@47”	~353”	**buy**	**360”**	**or**	**30’**	**cord**		
hemispheres				**buy**	**2**	**hemi-**	**spheres**			

Inventory for pieces of 4 transition models: pipe, slotted pipe, elbows, 3-ways, cord, and hemispheres.