Shiva: Two Views of Burnside's Lemma at Work

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Abstract

This paper documents the permutational development of artist James Mai's painting *Shiva: World Ages* and discusses two ways Burnside's Lemma can be seen to be at work in this painting, including which way is more appropriate. Ultimately, the authors examine the impact of the investigation on the artist's prior assumptions about the arc-forms as a set, and how new realizations about their relationships contribute to both the compositional organization and the metaphoric references indicated by the painting's title.

1. The Prehistory of *Shiva: World Ages*

The formal elements and organization of the painting, *Shiva: World Ages*, originated in an investigation of arrangements of four quarter-circle arcs arranged in quadrants of a square. In the initial investigation, each quarter-circle arc was capable of undergoing two different rotations and two different reflections, yielding four possible arcs to choose from in each of the four quadrants of the square. The artist's goal was to generate a complete set of distinct arc-forms up to symmetry, consisting of one arc chosen from each quadrant and to include this complete set in a single painting. This involved not only discovering all arrangements of four of the 16 arcs, but eliminating any arrangement that repeated another by reflection or rotation. All 16 such arcs are shown in Figure 1a, and two possible arc-forms are shown in Figure 1b. Through a process of diagramming and trial-and-error, it became clear to Mai that the set of distinct arc-forms would be too large and unwieldy for inclusion in a single painting. Since there are four arcs to choose from in each quadrant, there are a total of 256 arc-forms from which the set of visually distinct arc-forms would have to be culled.

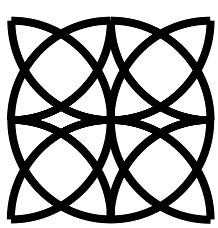


Figure 1a: *The 16-Arc Parent-form*

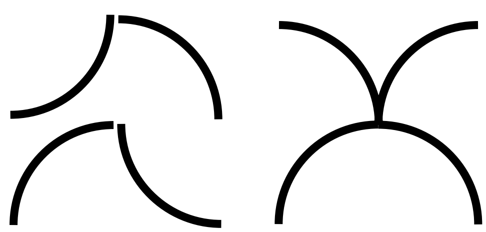


Figure 1b: *Two Arc-forms Created from the Parent with One Arc from Each Quadrant*

Indeed, had this artistic investigation been done after the artist became acquainted with Burnside's Lemma as it was used to analyze the forms in Mai's paintings *Permutations: Astral* and *Permutations: Earthly*, the result could have been used here too [1]. Rather than fully develop this technique again, we shall provide an overview. Whenever we are faced with the task of finding the number of visually distinct sub-forms up to symmetry in a set of all variations of a particular type of sub-form from some parent-form, such as Figure 1a, we may apply a version of Burnside's Lemma tailored to this particular type of artistic investigation. In order to make this statement clear, we need some preliminary definitions.

Every finite geometric form has a group of symmetries; this group may consist of only the identity, or do-nothing symmetry. These symmetries are the reflections and rotations that fix the figure. In other words, if the original figure is reflected or rotated by one of its symmetries, we could not see the difference. For example, the group of symmetries of Mai's form in Figure 1a is relatively simple; it consists of the eight symmetries of a square, which are four reflections and four rotations. The four lines of reflection, one vertical, one horizontal, and two diagonal, are seen superimposed on the 16-arc parent-form in Figure 2. The four rotations are of 0º, 90º, 180º, and 270º. For technical reasons we think of the identity symmetry as the 0º rotation. This group is known as $\boldsymbol{D}_4$ in the notation of group theory.

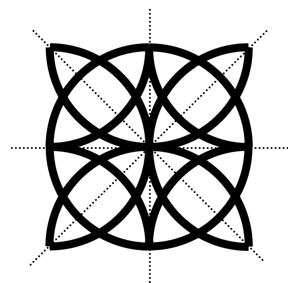


Figure 2: *The Parent-form with Lines of Reflection*

Each symmetry leaves a number, sometimes zero, of the individual sub-forms unchanged just as it leaves the parent-form unchanged. For example, the vertical reflection leaves not only the parent in Figure 1a unchanged, but also the right-most arc-form in Figure 1b. Note that none of the symmetries of the parent leaves the left-hand arc-form in Figure 1b fixed. The set of individual sub-forms that each symmetry leaves unchanged is called the *fix* of that symmetry. We use lower case Greek letters to name the symmetries. We will also use the following notation; the fix of a particular symmetry φ is fix(φ), and |fix(φ)| represents the number of sub-forms fixed by that symmetry. We will call the group of all symmetries of the parent-form G and let |G| indicate the number of symmetries in G, including the identity symmetry. With this in place, we may now state our tailor-made version of Burnside's Theorem.

If G is a finite group of symmetries of a parent-form S then the number of visually distinct figures in the family of sub-forms from S is given by $\frac{1}{|\mathrm{G}|}\sum_{\varphi\in\mathrm{G}}\left|\mathrm{fix}(\varphi)\right|$ *where this sum is taken over each symmetry in the group G.*

In other words, all we need do is find how many of the arc-forms in our family are fixed by each of the eight symmetries of the parent form, sum these, and divide by the total by eight. It is fairly straightforward to confirm that there are 43 visually distinct arc-forms that can be created by choosing one arc from each quadrant of Mai's 16-arc parent-form. This information allows the artist to know when the search for visually distinct arc-forms is complete, but yields no information about how to go about finding this smaller set of 43 forms amongst the larger set of 256 possible arc-forms. Hence, the artist's instincts were correct; this set is rather unwieldy. At this juncture in the investigation, Mai decided to refine his definition of arc-forms to obtain a smaller, usable set; however, work on this larger set continues.

2. Reframing the Investigation

Mai refined the parameters for generating the arc-forms with the goal of narrowing the resultant number of sub-forms. To this end, he limited the arcs available for any given form by permitting the use of each of the four orientations of the arc only once; see Figure 3. After one of the four orientations was employed in a quadrant, it could not be repeated in any other quadrant. A simple example of an arc-form

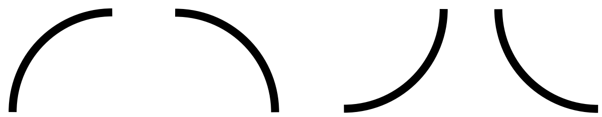


Figure 3: *The Four Possible Arc Orientations*

that obeys this new restriction is the circle, whose four component arcs are the four possible orientations. Note that neither of the arc-forms in Figure 1b would be allowed in Mai's set of forms under this new restriction. In fact, the restricted set of arc-forms now under investigation is just all the arrangements of the four quarter-arcs of the circle up to symmetry. Thus, Mai proceeded to search for these distinct forms by starting with a quartered circle and permuting its four arcs within the four quadrants, initially as exchanges of single pairs, then as pairs of pairs, and finally as cyclic permutations involving three arcs. The resulting set, after elimination of forms that repeat another by rotation, reflection, or combinations these, consists of eight distinct arc-forms, which is a considerably smaller and more manageable set than the first process would have produced. Certainly, the set is small enough for the artist to deal with on a trial-and-error basis. The eight forms and *Shiva: World Ages*, are displayed in Figure 4.

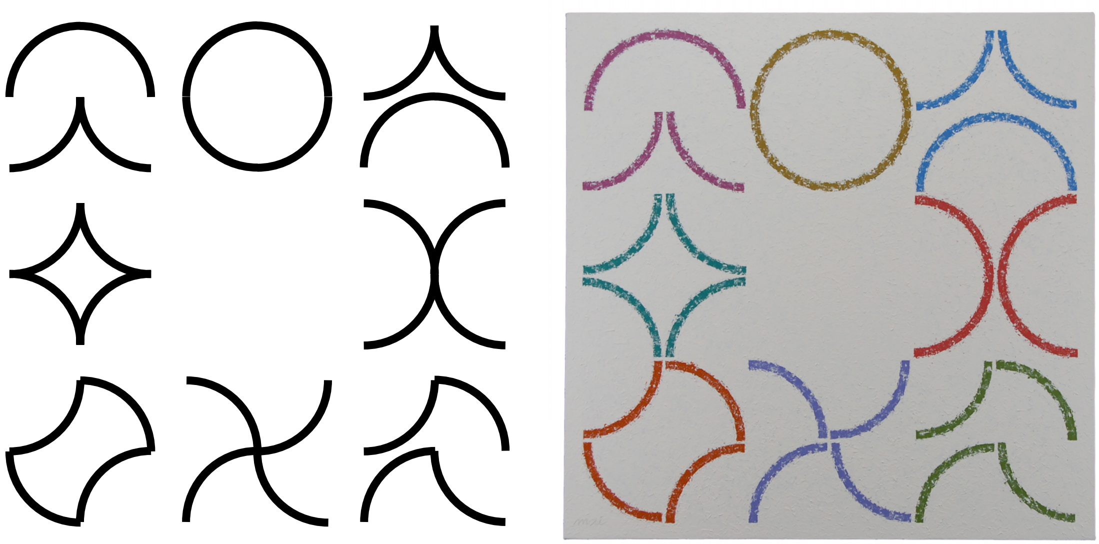


Figure 4: *The Eight Arc-forms and "Shiva: World Ages"*

Actually, we could still use our tailored version of Burnside as stated in Section 1 even in the case of the restricted investigation. The individual symmetries, and thus the symmetry group, remain the same, but the total number of possible sub-forms is greatly reduced to a set of 24 arc-forms from which to cull those fixed by each symmetry. For instance, the right-most arc-form in Figure 1b would be counted in the fix of the vertical reflection in the unrestricted investigation, but not in the restricted investigation because that form repeats arcs of the same orientation. That arc-form is still fixed by the vertical reflection, but it is not a valid arc-form in the restricted investigation. Again, it is relatively straightforward to conclude, via this altered version of Burnside's Lemma, that there must be exactly eight visually unique arc-forms that are built by choosing one arc from each quadrant with the restriction that each of the four possible orientations is used exactly once. Readers who are interested in a detailed analysis of using Burnside to count the number of sub-forms of a given type within a more complex parent-form should consult [1].

The problem with using Burnside in this manner is that it is not an honest mathematical analysis of Mai's process in this case. When creating *Shiva: World Ages*, Mai was no longer thinking of sub-forms within the original 16-arc parent-form of Figure 1a. He was thinking in terms of physically permuting the four quarter-circle arcs within the four quadrants. Thus, to bring our tailored version of Burnside to bear on the mathematical question of why there are exactly eight arc-forms is to superimpose onto the painting a mathematical structure that had been discarded by the artist as irrelevant or, worse, to introduce a mathematical structure that is outside the artist's creative purposes. To be sure, additional mathematical structure is sometimes needed to explain the geometric artifacts of art, but this technique should be avoided whenever possible. This is especially important in this case, since we can mathematically explain why there are exactly eight arc-forms without using this specialized version of Burnside.

There is further artistic evidence that applying the same type of mathematical analysis to *Shiva: World Ages* as was used on Mai's two *Permutations* paintings is disingenuous. A common feature of all three works is that the complete sets of forms that all three paintings enumerate are arranged, radially or in strata, about the center of each canvas. In each of the *Permutations* paintings the center displays its parent-form, the key to unlocking the content of the work. The center portion in *Shiva: World Ages* is conspicuously void; this absence of the parent-form declares a distinct difference in the context of the set of arc-forms from the set of forms in each of the *Permutations* paintings. The parent-form is most clearly visible in reproduction in *Permutations: Earthly* seen in Figure 5. Mai's compositional choice to leave the center of *Shiva: World Ages* vacant affirms his intent to focus on physical permutations of the four quarter-circle arcs and away from the individual forms as sub-forms of some parent. Interestingly, our decision to link the mathematical analysis to the compositional choices of the artist will lead us back to Burnside's Lemma, this time in its traditional algebraic form.

Figure 5: *"Permutations: Earthly", 42 x 42" (square), acrylic on canvas*

3. Permutations, Group Actions, and Orbits

Following the lead of the artist, we will mathematically associate each arc-form with a permutation of four symbols. Mathematicians refer to the group of all such permutations as $\boldsymbol{S}_4$, and so each possible permutation of the four quarter-arcs of a circle within a 2 x 2 grid can be named by a cycle. For example, the cycle (2 3 4) represents leaving the first arc untouched, placing the second arc the quadrant occupied by the third arc, moving the third arc to the quadrant of the fourth arc, and finally placing the fourth arc into the empty position absented by the second arc. The permutation (1 3)(2 4) represents swapping the positions of the first and third arcs and the second and fourth arcs. We denote the permutation where nothing is moved by (1). Note that these are rigid movements; there is no reflecting or rotating of arcs. Thus, this notation is only meaningful once we choose an initial position for each of the arcs and order them. Again, following Mai's original conceptualization, we will use the circle as our original configuration and number the arcs as shown in Figure 6a. Hence the permutations (2 3 4) and (1 3)(2 4) would produce the arc-forms shown in Figure 6b. We will leave a slight gap between the arcs so that each one is distinct from the others.

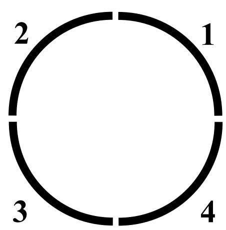


Figure 6a: *The Initial Arc Configuration*

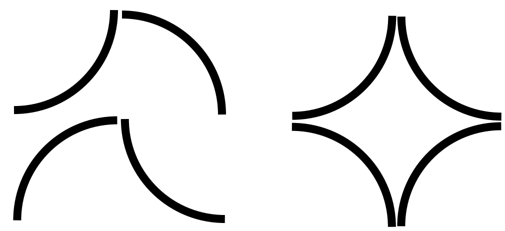


Figure 6b: *The Arc-forms of (2 3 4) and (1 3)(2 4)*

Now we must turn our attention to mathematically describing the artist's elimination of redundant arc-forms that are merely symmetric versions of another form. The symmetries that we need to take into account are just those that we have previously discussed, the symmetries of the underlying 2 x 2 grid. What we want to do is distribute the 24 possible arc-forms created by all possible permutations of the quarter-circles into subsets so that two forms are in the same subset if and only if they are symmetric copies of one another. For instance, the permutation (1 3 2) creates a symmetric copy of the arc-form created by (2 3 4), they are merely horizontal reflections of each other. This is demonstrated in Figure 7.

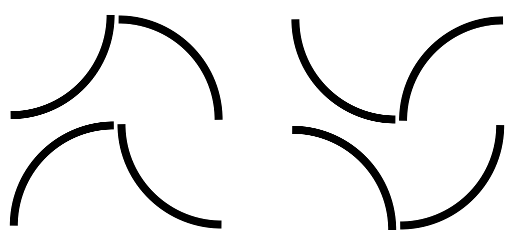


Figure 7: *The Arc-forms of (2 3 4) and (1 3 2)*

If we apply any one of our symmetries to all of the 24 arc-forms, we say that the symmetry is acting on the set of the forms. Since we are interested in the effects of all eight of our symmetries, we are interested in the action of the symmetry group on the set of arc-forms, hence the term group action. We can find all the symmetric copies of any particular arc-form by applying each of the eight symmetries, one at a time, and observing which of the other 23 arc-forms appear. The algebraic language for the set of arc-forms created by this process is the orbit of the initial form under the action of the symmetry group. This process does not rely on visual experimentation alone, because our group of symmetries, $\boldsymbol{D}_4$, can actually be interpreted as a subgroup within $\boldsymbol{S}_4$, so each symmetry can be expressed as a permutation. For instance, the 90º rotation can be expressed by the permutation **(1 2 3 4)** since it acts on the square by sending each vertex to the adjacent vertex. To distinguish permutations that represent arc-forms from permutations that represent symmetries, we will use bold faced type for the symmetries.

Table 1 shows each figure in the orbit of the arc-form named by the permutation (2 3 4), the permutation name of each form in the orbit, the symmetry that created each form from (2 3 4), and the permutation name of the symmetry. Note that the orbit of the arc-form from the permutation (1 3)(2 4) consists of just that form by itself. This is because no matter which of the eight symmetries we apply, we get the same form back. Interestingly, the action of the symmetry on the arc-form when both are expressed as permutations is not the normal multiplication of permutations, but rather the action is conjugation. So, the action of the symmetry **(1 2 3 4)** on the arc-form (2 3 4) is compute in the following manner.

$$\begin{aligned}\mathbf{(1\ 2\ 3\ 4)}[(2\ 3\ 4)] &= (1\ 2\ 3\ 4)(2\ 3\ 4)(1\ 2\ 3\ 4)^{-1} \\ &= (1\ 2\ 3\ 4)(2\ 3\ 4)(1\ 4\ 3\ 2) \\ &= (1\ 3\ 4)\end{aligned}$$

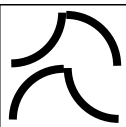
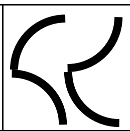
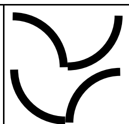
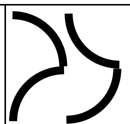
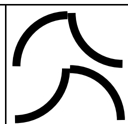
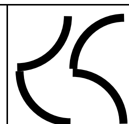
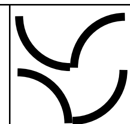
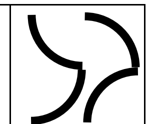
Arc-form								
Arc-form Permutation	(2 3 4)	(1 3 4)	(1 2 4)	(1 2 3)	(1 4 3)	(1 4 2)	(1 3 4)	(2 4 3)
Symmetry	0º Rotation	90º Rotation	180º Rotation	270º Rotation	Vertical Reflection	Left Diag. Reflection	Horizontal Reflection	Rt. Diag. Reflection
Symmetry Permution	**(1)**	**(1 2 3 4)**	**(1 3)(2 4)**	**(1 4 3 2)**	**(1 2)(3 4)**	**(1 3)**	**(1 4)(2 3)**	**(2 4)**

Table 1: *The Orbit of the Arc-form (2 3 4)*

Orbits have several interesting properties, which we will state here in the context of this investigation of arc-forms. Each form is in its own orbit because every form is a symmetric copy of itself. This implies that every form is in at least one orbit. If a particular form is in the orbit of another, then both of those forms generate the same orbit because each is a symmetric copy of the other, so any form that is a symmetric copy of the first is also a symmetric copy of the second and vice versa. This implies that any arc-form in a particular orbit can be used to generate that orbit. Thus, not only is each arc-form in at least one orbit, but each form is in exactly one orbit since if one form were in two orbits it could be used to generate both of those orbits; so the orbits would be identical, the same orbit. Therefore, Mai's quest is to partition the set of all 24 possible permutations of the quarter-circles into their orbits under the action of our symmetry group and choose one arc-form from each orbit to create his complete set of visually distinct forms.

Hence, the mathematical translation of Mai's investigation is that the arc-forms in *Shiva: World Ages* can be thought of as the set of orbits in $\boldsymbol{S}_4$ under the action of conjugation by the elements of $\boldsymbol{D}_4$. Now is when our mathematical investigation comes full-circle because the purely algebraic purpose of Burnside's Lemma is precisely to count the number of orbits a set is partitioned into by the action of a group. We now restate the result in its group theoretic form in the finite case.

Given a finite group G acting on a finite set S, the number of orbits that G partitions S into is given by $\frac{1}{|\mathrm{G}|}\sum_{\varphi\in \mathrm{G}}|\mathrm{fix}(\varphi)|$

Thus, mathematically we are back to where we started, but this time we have arrived by staying true to the artist's intent. For a proof of Burnside's Lemma see *Contemporary Abstract Algebra* by Joseph Gallian, any edition after the third.

4. Re-visioning the Circle and the Final Composition

In the final compositional stage, Mai worked to encode structural characteristics of the arc-forms through the use of color, scale, and distribution. To emphasize the fact that each form is a rearrangement of the same arc components, Mai painted all of the forms to the same scale. Since there are eight forms and Mai prefers to work within a square format whenever possible, the final composition was organized as a 3 x 3 square grid, with the eight forms occupying the eight peripheral units of the grid. In such an arrangement of close proximity and equal scale, the eight forms are difficult to distinguish visually from each other when rendered the same color; consequently, each arc-form is painted a different color to assist a quick visual recognition within the artist's compositional requirements.

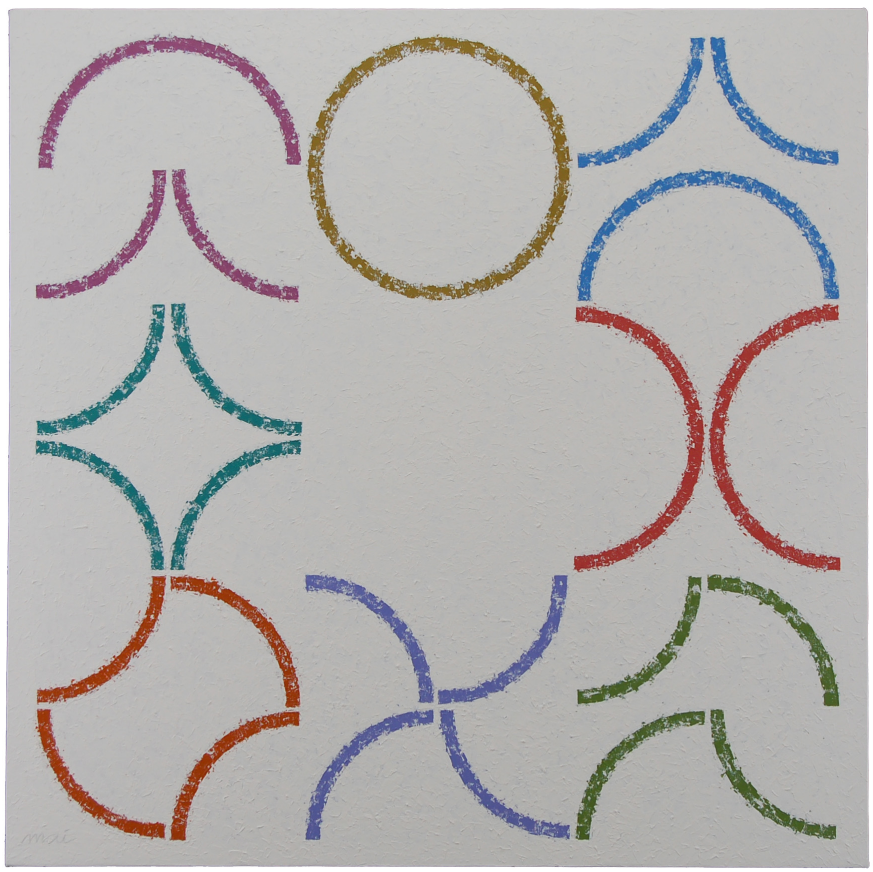


Figure 8: *"Shiva: World Ages," acrylic on canvas, 32 x 32"*

For Mai, one of the most important results of the painting was the recasting of his assumptions and expectations about the arc-forms themselves. When Mai began the investigation, he assumed the circle to be the basic, generating form from which all other arc-forms are derived as rearrangements of its four arcs. That hierarchy, of course, does not exist objectively in the relationships between the arc-forms themselves, but only in the artist's subjective choice of the circle as the point of departure. In fact, any arc-form may be considered as the base form from which all other forms in the set may be generated.

Hence, all forms are derivable from any one of the forms; they are variations without hierarchy. To be sure, there are perceptual qualities, such as maximal symmetry, closedness of shape, familiarity as a geometric object, and more, that might lead most viewers to posit the circle as the generating shape; however, nothing in the structure of the system lifts one form above the others. This changes one's understanding of the circle from an isolated object to a stage in a process of transformation, and so shifts the viewer's aesthetic understanding of the painting from the static to the dynamic, from the visual "noun" to the visual "verb."

In both the original and the restricted investigation, there is no objective hierarchy to the complete set of arc-forms. However, it should be noted that the original investigation would not have lead to the same type of artistic re-contextualization of the circle. In the first setting, the circle is merely one sub-form of the parent-form in Figure 1a. No form constructed by choosing one of the four arcs from each quadrant can be seen as generating all of the others; they are each static slices of the same parent-form. As in both *Permutations* paintings, only the parent-form can unlock the true content of the work. Thus, *Shiva: World Ages* is significantly different in that each individual arc-form carries within itself all the information needed to understand the complete set of forms and their relationships to each other.

This shift of consciousness is also reflected in the title, *Shiva: World Ages*. To the artist, the process of taking apart and recombining the four component arcs seemed like a series of destructions and creations. Mai was reminded of Hindu sculptures of Shiva Nataraja, which depict the god dancing the world into and out of existence, his body surrounded by a circle of fires representing the infinite cycle of world destructions and creations. With this reference in mind, Mai organized the eight arc-forms around the perimeter of the square canvas, with an empty space in the middle. This abstracted allusion to the compositional and symbolic elements of Shiva Nataraja helps to reinforce the consideration of the circle or any of the other forms as "one among many" and part of an endless dynamic cycle. Shiva humbles humanity with the realization that this universe, seemingly vast and ultimate, is only one of many that have come before and will come hence. By re-contextualizing our view of the world in this way, assumptions are recast and consciousness is changed.

This association with Shiva is not an incidental one for the artist. Mai employs systemic and mathematical procedures for similar purposes: to question assumptions and uncover new perspectives. New subjective insights are sought through objective investigations. The artist could easily have generated an open-ended set of arbitrary variations of arc-forms for use in a painting, but it was the exhaustive, complete nature of the permutational process that led him to reconsider the relationship between the circle and the other forms. The small shift of consideration from hierarchy to equivalence, from discontinuity to continuity among the forms is not a trivial matter for an artist who seeks to reshape his own perspectives through his art. Mai's commitment to the relevance of mathematics to art is rooted in its potential to open new creative considerations that other processes may not.

References

[1] J. Mai and D. Zielinski. Permuting Heaven and Earth: Painted Expressions of Burnside's Theorem. In *Conference Proceedings of Bridges: Mathematical Connections in Art, Music and Science, 2004*. Eds. Sarhangi, Reza and Sequin, Carlo.