Frieze Patterns of the Alhambra

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Abstract

A main feature of Islamic art is the use of infinitely-repeating geometric figures to cover planar surfaces. Some form two-dimensional "wallpaper" patterns and others appear in one-dimensional frieze patterns. The walls and ceilings of the palaces of the Alhambra contain an abundance of original and restored examples of both types of geometric ornament that are recognizably Islamic in character. This paper will consider only frieze patterns, presenting and analyzing a few beautiful examples from the Alhambra.

Introduction

People of every known human society have used *tilings* (defined here as edge-to-edge coverings of a surface such that no overlaps or gaps exist between the tiles) and *patterns* (defined as designs containing a systematically repeating *motif*) to cover the floors and walls of their homes from even the earliest of times. In Islamic decoration, a main characteristic is the use of a finite number of tile shapes to create intricate, infinitely-repeating geometric designs. The most famous building in the world for these sorts of examples is the Alhambra in Granada, Spain. The symmetries of the patterns comprising the ornamentation there may be classified as belonging to one of the 17 "wallpaper" groups (also known as plane crystallographic groups) for two-dimensional patterns, or to one of the 7 "strip" (or frieze) groups for one-dimensional patterns. We will confine ourselves to the latter type of periodic patterns found at the Alhambra that are composed of wood, glazed tile, or *alicatado* (Spanish for cut tiles, derived from the Arab verb *qata'a*, "to cut"), or *yeso* (plaster).

Classifying Frieze Patterns

Many important properties of decorative art depend upon the idea of symmetry. There are four rigid motions (or distance-preserving transformations, called *isometries*) of the plane. (For a nice, easy-to-follow proof of this, please see Appendix 1 in [1].) These include translations (movement in a given direction and through a given distance), rotations about a point (the center of rotation), through a given angle (the angle of rotation), reflections in (or across) a line, (the mirror line or axis of reflection), and glide-reflections in which reflection in a line (the glide mirror) is combined with a translation through a given distance and parallel to the mirror line. For rotations, any point about which a pattern may be rotated through the angle $360°/n$ leaving the pattern invariant, is said to be a center of n-fold rotational symmetry. Hence, patterns with symmetry are those that have at least one of these four distance-preserving transformations leaving them invariant.

Periodic frieze patterns extend in only one direction, along a strip, and are invariant under a translation which we assume is of positive minimum length. For classifying these, we will assume (in theory) that the design motif repeats infinitely in a horizontal direction and is uncolored. Using mathematical group theory, we find that there are only seven different possible groups of symmetries for such one-dimensional patterns. (A nice proof of this may be found in Appendix 2 of [1]). Each *symmetry group* is the collection of all isometries that leave a frieze pattern invariant. The generally accepted way to describe these seven groups uses a four-symbol notation, ***pxyz,*** known as the *international symbol*. The initial letter, ***p*** (which indicates that the motif may be contained within a primitive cell), is followed by either the symbol ***m*** (indicating the existence of reflection symmetry with mirror at right angles to the translation axis) or a "**1**" (indicating the lack of vertical mirror reflection symmetry). The third symbol may be an "**m**" (indicating the existence of reflection symmetry with mirror parallel to the horizontal translation axis), a "**1**" (indicating the lack of such a mirror reflection), or a "**g**" (indicating the existence of glide-reflection symmetry with glide mirror parallel to the translation axis and no horizontal mirror reflection symmetry). The fourth symbol may be either a "1" or a "2" indicating the *n*-fold rotational symmetry. For patterns with no rotational symmetry, ***n* = 1**, and patterns with 2-fold rotational symmetry have ***n* = 2** as their fourth symbol. For a more in-depth discussion, please consult [2]. By definition, all of the seven periodic frieze pattern types have translational symmetry. Using the notational rules just explained, here are the symbolic notations for the seven symmetry groups of periodic frieze patterns: ***p*111, *p*112, *pm*11, *p*1*m*1, *pmm*2, *pmg*2, and *p*1*g*1**. Frequently the four-symbol notation is shortened to two-symbol form; here are the corresponding group symbols in the same order: **11, 12, *m*1, 1*m*, *mm*, *mg*, and 1*g*.** On the following pages, each frieze type will be described more fully and an illustrative example from the Alhambra given, showing either a repeat unit (that produces the frieze pattern via translation only) or a primitive cell (which is the smallest portion of a motif that produces the pattern using all of the isometries in its symmetry group); rotation centers and mirrors are also indicated.

Analyzing Frieze Patterns

To analyze and classify one-dimensional periodic frieze patterns, one needs to be able to recognize the existence of the symmetries of each pattern, that is, the isometries that leave the pattern invariant. This author looks for obvious mirror reflections first and then attempts to ascertain the rotational symmetry next. Determining whether or not a pattern has glide-reflection symmetry can be challenging since many times these may be subtle. Let's consider the seven classifications one at a time.

*p*111 Frieze Patterns

A classification of ***p*111** indicates a pattern has no rotations, reflections, or glide-reflections as symmetries. For example, the strip of lowercase letters "b" below may be continued indefinitely as we translate it repeatedly to the left and to the right. Note that there is no rotation or reflection symmetry present.

… b b b b b b b b …

An example of such a pattern in the Alhambra is a calligraphic inscription found at the summit of a pillar in the *Mexuar* room. The Arabic script forming the motif (boxed by line segments in Figure 1 on the next page), may be translated as "There is no victor but Allah." This saying, a well-known motto of Sultan Muhammad I and his *Nasrid* successors, is found throughout the Alhambra. Notice that the motif is asymmetric, that is, it has no rotation or reflection symmetry.

Figure 1. A glazed tile calligraphic inscription with a possible repeat unit (outlined) for a *p*111 pattern having only translation symmetry.

*p*112 Frieze Patterns

A ***p*112** pattern has two-fold rotational symmetry, but no reflections (with mirrors either perpendicular or parallel to the translation axis) nor any glide-reflections as symmetries. For example, starting from the left in the strip below, the lowercase letter "b" may be rotated 180 degrees about a point on the translation axis to form the second lowercase letter, "q." The "q" may then be rotated once again to form a second letter "b," and so on, as we progress towards the right. (This may also be repeated to the left). In the pattern, the rotation centers are all on the "midline" of the pattern, and are equi-spaced.

… b q b q b q b q …

Figure 2 shows a ***p*112** frieze in the Alhambra; a glazed tile showing an interlaced knotwork design**.** The pattern's motif is two different interlaced forms, each having a two-fold rotational symmetry, but no reflection or glide-reflection symmetry. A primitive cell containing half of each form is outlined by line segments; the two-fold centers of rotation are surrounded by circles. Repeated rotation of the primitive cell about these centers produces the frieze pattern.

Figure 2. Part of a glazed tile dado (lower wall tiling) showing a primitive cell (outlined) of a *p*112 pattern. Rotation centers (encircled) lie at the midpoints of two edges of this cell.

*pm*11 Frieze Patterns

A horizontal ***pm*11** pattern has vertical mirror reflection symmetry but no rotation symmetry. For example, in the strip below, the lowercase letter "b" may be reflected across a vertical mirror axis, perpendicular to the translation axis to form the lowercase letter "d." The "d" may now be reflected in a similar manner to form the second "b," and so on, as we progress towards the right. (This may also be repeated to the left). In this pattern, the reflected pair "b d" is repeated at regular intervals by translation.

… b d b d b d b d …

An example of a ***pm*11** design is provided below**.** Half of the pattern's motif is contained within a primitive cell (outlined), and repeated reflections in vertical mirrors at its left and right edges produce the pattern.

Figure 3. Part of a glazed tile border showing a primitive cell (outlined) of a *pm*11 pattern with vertical mirrors at the left and right edges of the cell.

*p*1*m*1 Frieze Patterns

A ***p*1*m*1** pattern has reflection symmetry, with the mirror parallel to the translation axis, but has no rotation symmetry. For example, in the strip below, the lowercase letter "b" may be reflected across the translation axis to form the lowercase letter "p." This pairing of the "b" and the "p" may then be translated towards the right. (This may also be repeated to the left).

… b b b b b b b b …
… p p p p p p p p …

An example of a ***p*1*m*1** design is given in Figure 4. Half of the pattern's motif may be contained within a primitive cell (outlined), which has a mirror at its top edge, on the midline of the pattern. There is no rotational symmetry present. Reflection in the mirror and repeated translation produces the pattern.

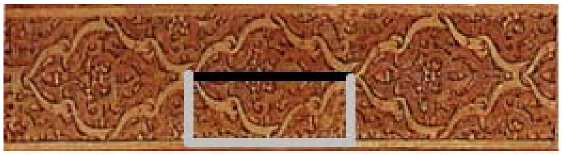

Figure 4. A *yeso* border showing a primitive cell (outlined) of a *p*1*m*1 pattern with mirror on the midline of the frieze.

*pmm*2 Frieze Patterns

A ***pmm*2** pattern has reflection symmetry in two directions (mirrors perpendicular and parallel to the translation axis), as well as two-fold rotational symmetry. For example, in the strip of letters below, the lowercase letter "b" is reflected across the translation axis to form the lowercase letter "p." Then both the "b" and the "p" are reflected across a mirror axis perpendicular to the translation axis to form the letters "d" and "q," respectively. A two-fold rotational center is created where the vertical mirror intersects the horizontal mirror—the rotation is a result of composing the vertical and horizontal reflections. The grouping of "b," "d," "p," and "q" is then translated repeatedly towards the right. (This may also be repeated to the left).

… b d b d b d b d …
… p q p q p q p q …

An example of a ***pmm*2** design in the Alhambra is given in Figure 5. One quarter of the pattern's motif is contained within a primitive cell bounded by mirror lines on three sides (two perpendicular and one parallel to the translation axis). Centers of two-fold rotational symmetry (encircled) are at the corners. Repeated reflection in the mirrors surrounding the primitive cell generates the whole frieze.

Figure 5. Part of a glazed tile dado showing a primitive cell (outlined) of a *pmm*2 pattern. Mirrors are along the top edge and on the vertical edges of the cell; the two centers of two-fold rotational symmetry are encircled.

*pmg*2 Frieze Patterns

A ***pmg*2** pattern has two-fold rotational symmetry as well as vertical reflection symmetry and glide-reflection symmetry along the axis of translation. For example, in the strip below, the lowercase letter "b" may be rotated about a point on the translation axis to form the second lowercase letter, "q." This pairing of "b" and "q" may then be reflected vertically to form the lowercase letters "p" and "d," respectively. Translating the four-letter block of "b q p d" repeatedly right (and left) produces the whole frieze. An alternate way to produce the pattern is to glide-reflect the "b" and "q" pair along the translation axis to produce the additional "p" and "d" pair in the four-letter block of "b q p d." Note also that there are two-fold rotational centers along the translation axis between the "p" and the "d."

… b q p d b q p d …

An example of a ***pmg*2** design in the Alhambra may be found as part of a glazed tile dado (or lower wall tiling) shown in Figure 6. There is two-fold rotational symmetry as well as vertical mirror reflection symmetry and glide-reflection symmetry along the translation axis. A primitive cell is outlined with a mirror on one vertical edge (indicated by a light line segment) and a two-fold rotational center (encircled) at the center of the other vertical edge. Note that in this frieze tiling, the background and foreground tiles have identical shapes. This property is often called "counterchange symmetry."

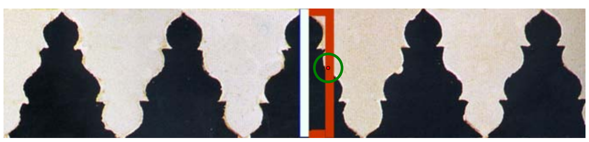

Figure 6. Part of a glazed tile dado of a *pmg*2 pattern, showing a primitive cell (outlined) with a vertical mirror (light line segment on the left) and a center of two-fold rotational symmetry (encircled).

*p*1*g*1 Frieze Patterns

A ***p*1*g*1** pattern has no rotation or reflection symmetry, but has glide-reflection symmetry along the translation axis. In the strip below, the lowercase letter "b" is glide-reflected (translated along the translation axis and then reflected across it) to form the lowercase letter "p." This same glide-reflection is repeated on the "p" to produce the next "b", and so on, to produce the frieze pattern.

… b p b p b p b p …

An example of a ***p*1*g*1** design in the Alhambra may be found as part of a *yeso* wall panel shown in Figure 7. A primitive cell has been outlined. Repeated glide-reflection of this cell vertically produces the whole frieze.

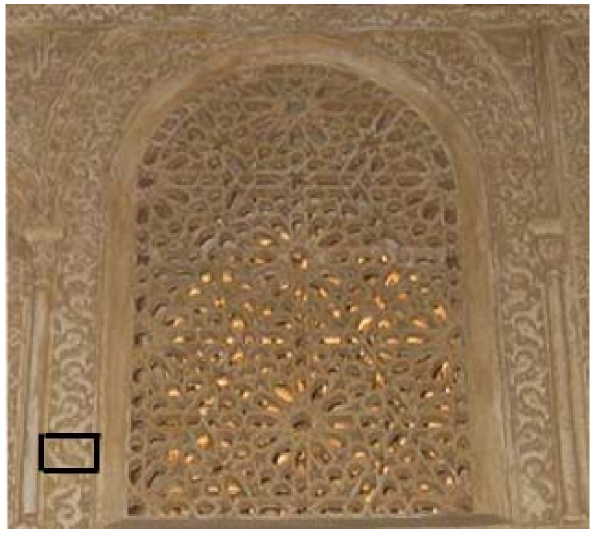

Figure 7. Part of a yeso wall panel showing a primitive cell (outlined) of a *p*1*g*1 frieze pattern.

Discussion

Given the surfeit of ornamentation on the walls of the Alhambra, one would expect to find at least a few examples of each of the seven possible frieze types. This author was able to find at least one of each, however, some of the pattern types were more abundance than others. For example, the ***p*1*g*1** class seems to appear very rarely in planar mosaic tilings. In *Symmetries of Islamic Geometric Patterns* [3], Abas and Salman provide a chart showing the statistical distribution of the various symmetry groups of the Islamic patterns they have compiled. Two of the three rarest categories of wallpaper patterns involve glide-reflections (*pg* and *pmg*), indicating that this isometry may be a less-favored symmetry for planar Islamic mosaic tilings. Thus, glide-reflections in frieze patterns may also be rare. When studying the tilings at the *Real Alcázar* in Seville, Spain (and as reported in [4] and [5]), this author was unable to find any examples of the ***p*1*g*1** class.

The author would like to thank the reviewers for their helpful comments and suggestions for improvement of this paper.

References

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