Mathematics and Music – Models and Morals

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Abstract

The intimate association between mathematics and music can be traced to the Greek culture. It is well-represented in the prevailing Western musical culture of the 18th and 19th centuries, where the traditional cycle of fifths provides a mathematical model for classical harmony that originated with the well-tempered, and later the equal-tempered, keyboard. Equal-temperament gives equivalent status to all twelve tonal centres in the chromatic scale, leading to a high degree of symmetry and an underlying group structure. This connection seems to endorse the Pythagorean concept of music as exemplifying an ideal mathematical harmony. This paper examines the relationship between abstract mathematics and music more critically, challenging the idealized view of music as rooted in pure mathematical relations and instead highlighting the significance of music as an association between form and meaning that is negotiated and pragmatic in nature. In passing, it illustrates how the complex and subtle relationship between mathematics and music can be investigated effectively using principles and techniques for interactive computer-based modelling [17] that in themselves may be seen as relating mathematics to the *art* of computing – a theme that is developed in a companion paper [2].

1. Mathematics and music in harmony

1.1. The music of the spheres. The notion of an intimate relationship between mathematics and music has a long history. The correlation between vibrating strings of different lengths and musical pitch established a strong connection between numerical and aural relationships, and supplied the basic intervals upon which the Western musical tradition is based. The vibration of a string could be identified as made up of many concurrent basic vibrations, comprising the fundamental note and many harmonics. The qualities of notes as played by different instruments could be analysed with respect to the *harmonics* they generated. The harmonic constituents of sounds acquired such fundamental importance in thinking about music that they later became the point of departure for many music theorists. Schenker (1868-1935) [11], for instance, set out to demonstrate that the tonal system had its roots in nature, and – building on this basis – even went so far as to attribute a primary, distinguished status to musical traditions based on tonality, and to their underlying aesthetics. In this view, the correspondence between the findings of natural science, mathematical models and human psychology gave a special significance to the classical music culture of the West.

1.2. The cycle of fifths. The advent and development of keyboard instruments led to a rationalization that had enormous ramifications for music. Once the range of a keyboard was sufficient to allow a single interval to be spanned both by a sequence of octaves and "perfect fifths", the need for some compromise in tuning each octave span of the instrument became apparent. This led to a mode of tuning known as "well-temperament", first widely used in the Baroque period, that had the liberating side-effect of allowing any note on the keyboard to be the tonal centre of a musical composition – a development celebrated to great effect by J S Bach in his 48 Preludes and Fugues for the Well-Tempered Clavier. The influence of composers such as Bach on musical composition was profound. It opened the way to a full exploration of tonal possibilities associated with all twelve notes of the chromatic scale. Where a mathematical perspective on music is concerned, there is a subtle but significant distinction between *well* temperament and *equal* temperament, as practiced from the middle of the nineteenth century. To quote Loy's account of tempering [8]: with well-temperament: "None of the scales or chords sounded bad. In fact every major and minor key sounded different. C sounded placid and fairly uninteresting. The more distant keys sounded more interesting. You might call some keys harsh, or agitated, or tense. And so, music could be written to suit the mood (or color) of each key". With equal temperament, in contrast: "No key sounds bad. No key sounds pure. All keys are somewhat interesting. But, they are all the same".

In effect, the emancipation of all twelve major and minor keys made it possible to conceive key relationships in an abstract mathematical fashion. The most widely cited form of this structure is that based on the cycle of fifths, as depicted in the model shown at the left in Figure 1. [The figures in this paper have been extracted from a poster for which a full colour image can be accessed and downloaded from http://empublic.dcs.warwick.ac.uk/projects/kaleidoscopeBeynon2005/posters/erlkonigPoster.pdf] In the cycle of fifths, adjacency of keys is associated with similarity of key signature. This connects keys whose tonic notes differ by a perfect fifth (cf. the red edges making up the two circuits in Figure 1) and connects a major key with its "relative minor" key with which it shares the same key signature (cf. the bi-directed green edges that define the spokes). As in Figure 1, capitalized and lower case letters will be used to refer to major and minor keys respectively.

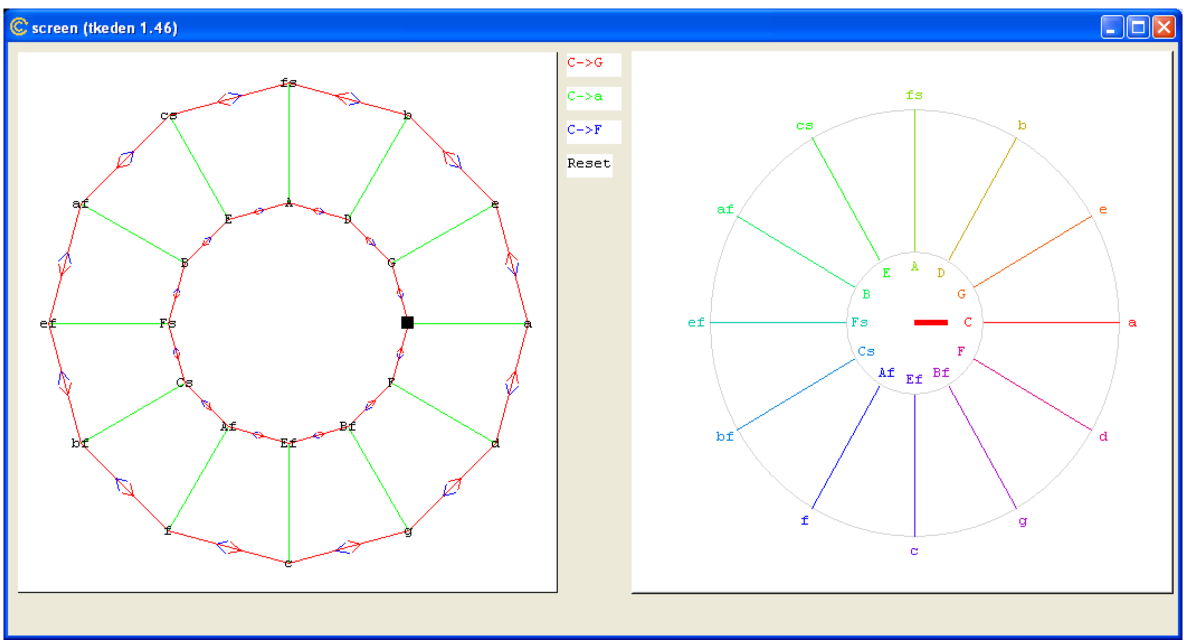
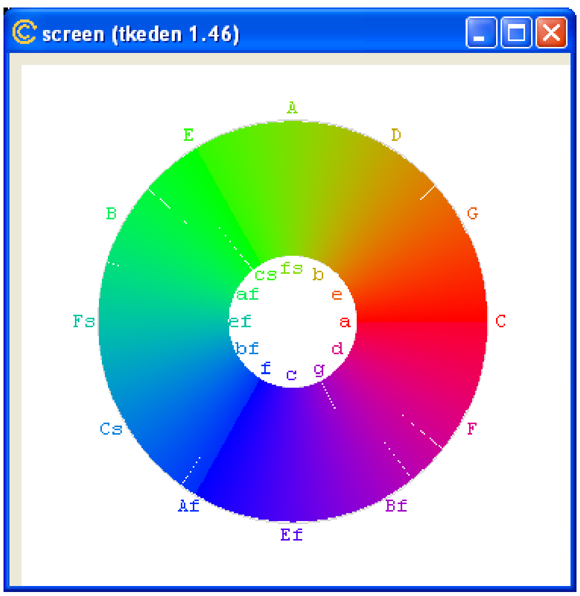


Figure 1: *The cycle of keys and an associated colour wheel*

1.3. Modelling the cycle of keys. Figure 1 depicts an interactive computer model of the cycle of keys that has been constructed by the author using modelling principles and tools that have been described in detail elsewhere [2,17]. The model comprises two connected visual components to represent two different ways in which the notion of current key is perceived in the mainstream tradition of tonal music. The diagram on the left (a "group graph") is the kind of model that underlies the basic music theory that is required to learn skills such as playing scales on an instrument. It is also used for harmonic analysis in music, as it might be applied to the works (e.g.) of Bach, Handel, Haydn, Mozart, Beethoven, Schubert, to traditional hymn tunes and to certain idioms of popular music. It is a framework that is harder to apply directly to music from the so-called 'romantic' tradition of the nineteenth century, as represented in the music of (e.g.) Chopin, Wagner, Liszt and – to a lesser extent – Schumann or Brahms, because of the greater complexity of their harmonies, and is a framework that does not apply to music of earlier traditions (such as that of Monteverdi or Palestrina) or to 20th century music that is atonal or polytonal.

At the heart of the harmonic system is the cycle of keys depicted in the group graph on the LHS of Figure 1. There are twelve such keys, corresponding to the 12 essentially distinct notes C, $C^{\#}$, D, $D^{\#}$, E, F, $F^{\#}$, G, $G^{\#}$, A, $A^{\#}$, B on a piano keyboard. In the group graph, each node on the inner cycle represents the tonic note in a specific major key, and this is connected by a green bi-directed edge to the submediant – the tonic note of its relative minor, by a red edge to the dominant, and by a blue edge to the subdominant. In this context, the submediant, dominant and subdominant respectively refer to the notes that lie at intervals of a minor third below, a fifth above and a fourth above the tonic note respectively. In Figure 1, the current tonic note is indicated by a small black square, and corresponds to the key of C, with relative minor a, dominant G and subdominant F. The current key is also indicated in the display on the RHS of Figure 1, in which the keys are disposed on the spokes of a wheel and coloured so as to suggest the different ways in which a musician experiences keys. As a mathematical model, the group graph in Figure 1 is associated with a particular Abelian product of cyclic groups, namely $C_2 * C_{12}$. The symmetry of the group graph reflects the sense in which an experienced musician maintains a sense of the global tonal framework whatever the current key.




```
func colourwheel {
        para angle;
        auto r0, g0, b0;
        if ((4*PI/3<=angle)&&(angle<2*PI)) {
                angle = angle - 4*PI/3;
                r0 = int(255*sin(2.0*angle/3.0));
                g0 = 0;
                b0 = int(255*cos(2.0*angle/3.0));
        }
        else if ((2*PI/3<=angle) && (angle<4*PI/3)) {
                angle = angle - 2*PI/3;
                r0 = 0;
                g0 = int(255*cos(2.0*angle/3.0));
                b0 = int(255*sin(2.0*angle/3.0));
        }
        else {
                r0 = int(255*cos(2.0*angle/3.0));
                g0 = int(255*sin(2.0*angle/3.0));
                b0 = 0;
        }
        return rgb2colour(r0,g0,b0);
}
```

Figure 2. *The colourwheel from which key colours have been extracted, together with its functional definition*

1.4. Modelling keys by colours. The component on the right-hand side of the model (the colourwheel) reflects a complementary aspect of musical experience: the sense that each key has its own distinct character and 'colour'. Notwithstanding the symmetry of the cycle of keys, it would (for example) be regarded as a significant act in musical terms to rewrite Beethoven's Choral Symphony (written in D minor) in C minor. One aspect of this is that the choice of key influences the absolute pitch and impacts on the technical difficulty and even feasibility of the instrumental parts. Over and above this, the associations of different keys are deeply embedded in the classical musical tradition, and appear to exercise an important influence over the kind of music that a particular composer writes.

The model depicted in Figure 1 exploits definitions similar to those underlying the cells of a spreadsheet to maintain dependencies between the values of observables. As an example of such a dependency, the colours associated with keys on the RHS of Figure 1 are drawn from the colourwheel in Figure 2, where the association between the colour of a ray and its orientation is defined by the piecewise function – based on interpolating between the red, green and blue colour constituents – that is encoded on the right of Figure 2. Another dependency is used to link the abstract change of key in the group graph (as typically recorded cerebrally by the musician in the role of musical analyst) to the associated experiential change in tonal colour (as typically registered as a 'felt experience' by the performer). In this way, changes of key – and to some extent ambiguity about current key – may be represented by the motion of a 'colour wheel' in which each direction corresponds to a different colour.

2. Some hints of discord

2.1. The status of mathematical models of music. Mathematical models of musical phenomena may appear to endorse the absolute, transcendental nature of the relation between mathematics and music. It is perhaps fitting that some of best illustrations of modulation conforming closely to the cycle of keys are to be found in the hymnals of the 19^{th} century. But, whilst the cycle of keys and the colourwheel can be interpreted as concretizations of precise abstract mathematical concept in the spirit of Turkle and Papert [15], the notion that they embody an absolute mathematical theory of music is suspect. From a detached musical standpoint, the colourwheel and group graph of Figure 1 may be seen as contrived mathematical abstractions within a concrete world of fuzzy experience.

This is most clearly evident in respect of the colourwheel. There is no accepted rationale to justify the choice of a particular key-colour association (whether or not a musician consciously associates keys with colours – as in instances of synaesthesia, to which composers such as Scriabin may have been subject [5]). In an extended discussion of this issue, Tovey [14:8,9] observes that "Beethoven ... when setting Scottish melodies, wrote to his English publisher, Thomson, that the key of A flat did not fit a certain tune that was sent him, inasmuch as that tune was marked *amoroso*, whereas the key of A flat should be called *barbaresco*", but goes on to deride the notion that "transposition would be equivalent to altering all the colours of a picture" as "a favourable example of the kind of fantasy which many learned musicians still fail to confine to its proper place among psychological obscurities".

Similar reservations apply to the group graph. In the first place, in aural terms, the cycle of keys is associated with the accidental approximation in pitch that underlies the notion of equal-temperament. Specifically, the ‘cycle’ is based on the false premise that 12 intervals of a perfect fifth (such as one might trace on a piano keyboard) generates exactly the same difference in pitch as a sequence of seven octaves. In reality, moving up a perfect fifth raises the frequency of a note by a factor of 1.5, whilst moving up an octave raises its frequency by a factor of 2, and $(1.5)^{12} = 129.746337890625$ merely approximates 2^7. On this basis, the formal model is an inexact abstraction from experience, in contrast to (say) the Platonic lines and points of Euclidean geometry. (Compare the rhythmic device used by Schumann in the second of the Drei Stücklein of his Bunte Blätter op. 99, where the 4^{rd} and 6^{th} notes in a bar comprising 8 demi-semiquavers carry a melody to be played in triplet time – a mathematical impossibility, but plausible in the specified *Sehr Rasch* tempo since 8/3 and 16/3 approximate 3 and 5 respectively.)

Secondly, the correspondence between the abstract harmonic framework depicted in Figure 1 and music within a particular tradition cannot be formalized. The closest correspondence is typically to be found in the least sophisticated music of Haydn or Mozart, where the harmonic structure is traced beat-by-beat through the chords of an accompaniment, and the changes of key such as are represented in the group graph predominate. In contrast, though Bach’s music (and the 48 Preludes and Fugues for the Well-Tempered Clavier in particular) is conceived within the framework, the characteristic texture of his music does not for the most part lend itself to a simple association between simultaneously sounding notes and tonal harmonies. And whilst most of the music of Beethoven and Schubert relies upon clear harmonic textures, it exploits tonal juxtapositions quite unlike those represented by the edges of the group graph in Figure 1. The problematic aspects of the model of key relationships in Figure 1 are exemplified in the distant relationship it establishes between tonic major and minor, of which Tovey [14:11] writes: "First, let us be quite clear about the contrast between tonic minor and tonic major. Remember that the contrast is not a 'modulation' or change of key at all: it is a change of outlook while we stay at home."

2.2. Alternative models of tonality. In his analysis of tonality in Beethoven [14], Tovey advocates an alternative model of tonality. The principle behind his analysis is that the keys that are most closely related harmonically to a major key are those whose triads can be found in its major scale. For example, applying this criterion, the key of C is directly related to d, e, F, G and a. A similar criterion can be applied to minor keys: for instance, the key of c is directly related to Eb, f, g, Ab and Bb. This supplies an alternative to the algebraic model of tonal relationships in the cycle of keys – a looser concept of neighbourliness of keys that can be expressed geometrically by laying out all 24 major and minor keys on the surface of a torus. Such a geometric framework was devised by Schoenberg [12, 10], and set out as a chart of harmonic regions as depicted in Figure 3.

			$d^{\#}$	$F^{\#}$	$f^{\#}$	A	a	C
			$g^{\#}$	B	b	D	d	F
			$c^{\#}$	E	e	G	g	Bb
	$d^{\#}$	$F^{\#}$	$f^{\#}$	A	a	C	c	Eb
	$g^{\#}$	B	b	D	d	F	f	Ab
	$c^{\#}$	E	e	G	g	Bb	bb	Db
$F^{\#}$	$f^{\#}$	A	a	C	c	Eb	eb	Gb
B	b	D	d	F	f	Ab		
E	e	G	g	Bb	bb	Db		
A	a	C	c	Eb	eb	Gb		
D	d	F	f	Ab				
G	g	Bb	bb	Db				
C	c	Eb	eb	Gb				

Figure 3. *The chart of the regions, from Arnold Schoenberg: Structural Functions of Harmony*

As explained in detail in [14], Tovey's treatment of tonality has a quite different focus from the cycle of keys model in Figure 1. Where the edges of the group graph purport to specify the particular modulations that normally arise in classical harmonic sequences, the emphasis in Tovey's analysis is rather on a more topological notion of neighbourliness of keys. For instance, whereas the group graph distinguishes modulations to the local relative major/minor, dominant and sub-dominant, Tovey maintains that these keys are no more closely related to the tonic than the rest of the five related keys. This accords with Tovey's central thesis about Beethoven's music – that it is not so much distinguished by novel local harmonic progressions, but by its contribution to "the long-range power of handling tonality [that] is in its earlier stages downright incompatible with concentration upon new chords and new progressions".

Further analysis of the music of the later classical period suggests other ways of elaborating on standard models of tonality. In addition to modulation to neighbouring keys in Figure 3, Tovey [14] also identifies Neapolitan transitions as characteristic of Beethoven's harmonic idiom – as exemplified in the initial bars of the Appassionata Piano Sonata op. 57, where the opening phrase is immediately repeated a semitone higher. The music of Schubert is also recognized as exhibiting novel harmonic effects, characterized, especially in the context of his finest songs, by unexpected dramatic progressions. The brief discussion that follows is elaborated in [2].

At one level, Schubert's music is characterized by very strongly defined local harmonic progressions, more in keeping with an algebraic than a topological model of structure. The simple ballad *Der König in Thule* is an excellent example of a musical composition that exemplifies structure within the traditional cycle of fifths framework. It not only exhibits direct modulations between closely related keys, but has the remarkable feature of being based solely on chords that are in first position – that is, that have their tonic note in the bass. Of Schubert, Tovey writes [14:30]: "He came only gradually under Beethoven's spell after his sense of tonality had developed quite independently; but his tonality is exactly Beethoven's in its fullest range, and is intensified by concentration in lyric forms to which Beethoven contributed little". Schubert's "independent sense of tonality" arguably put more primary emphasis on the traditionally most commonplace modulations (to the dominant, subdominant and relative minor), and in lyric forms focused on harmonic tension and argument rather than on loose and exploratory juxtaposition of keys. Through reinforcement of conventional expectation, such an idiom gives greater scope for immediate local harmonic surprise. To some listeners, it also conveys an unwelcome impression of naivety – witness George Bernard Shaw's scathing critique of the Great C Major Symphony: 'a more exasperatingly brainless composition was never set on paper' [13:99].

Because of the underlying orientation of Schubert's harmonic idiom, the restricted model of tonality afforded by the group graph in Figure 1 is a promising basis for the analysis of his music, but something further is required. Attempting to apply the simple techniques that apply to the *Der König in Thule* to more sophisticated compositions poses problems. One of Schubert's trademark harmonic innovations is found in his treatment of tonic major and minor. Much more is involved here than Tovey's "change of outlook whilst we stay at home". In the extended ballad *Erlkönig*, one of Schubert's most ambitious songs, it is not simply that the music visits tonic major and minor keys in quick succession, but that tonic major and minor are conflated in passages in such a way that the whole tonality of the music is called into question. Figure 4 depicts two experimental models, each constructed by modifying the group graph in Figure 1, that offer alternative ways of expressing Schubert's use of tonic major-minor.

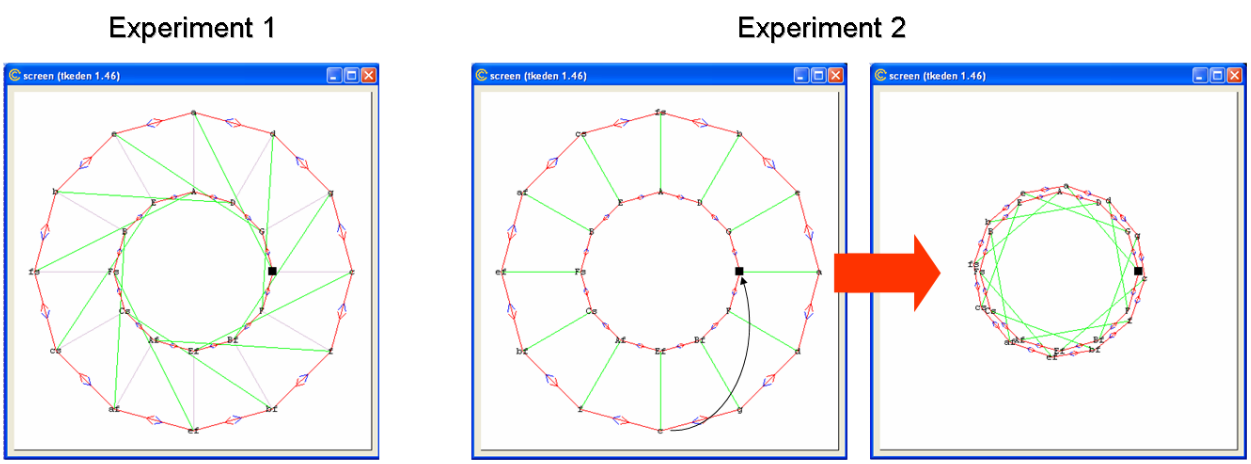


Figure 4. *Two experiments in modelling Schubert's conflation of tonic major and minor*

In Experiment 1, a new generator is introduced to represent tonic major-minor modulation. For this generator to be its own inverse, it is necessary to reappraise the structure of the group. This entails reversing the directed edges on the outer circuit. As is explored by Waller [16], this identifies the resulting group graph with the automorphism group of the toroidal space of keys depicted in Figure 3. In Experiment 2, the group graph in Figure 1 undergoes a different transformation, corresponding to being mapped on a homomorphic image in which tonic major and minor tonalities become conflated. This better reflects the character of Schubert's harmonic innovation.

3. Resolution through model making

3.1. Making mathematical models of music. The above discussion illustrates something of the range and diversity of issues that are involved in making models to assist musical analysis. It highlights the contrast between alternative ways of interpreting group graphs – as encoding distinguished transitions (as in the classical cycle of keys), or as defining a set of transformations in a space of tonality. These alternative views relate to algebraic and topological perspectives on tonal relationships. Waller's model [16], a variant of the group graph on the left of Figure 4 in which the group generators are the three involutions exemplified by C↔a, C↔c, C↔e, illustrates a further nuance: these three generators are implausible as distinguished modulations in a standard musical idiom, and have no special significance as generators of the automorphism group of the space of keys, but are informative in relation to symmetries and musical chords such as augmented triads and diminished sevenths. Equally important is the nature of the interpretation of mathematical diagrams involved. The virtue of Experiment 2 as a representation is that it accommodates the harmonic excursions of *Erlkönig* without dispensing with the cycle of fifths model of tonal relationships. In using this model in context (as when tracking the harmonic progressions as the song is played [2]), it is not appropriate to make sudden transitions between the full group graph and its homomorphic image. The gradual transformation of one group graph into the other suggested by the black arrow in Figure 4 is more convincing, and could be appropriately synchronised with the music so as to suggest the presentiment of encroaching harmonic disintegration that an appreciative and familiar listener experiences. Significantly, such an interpretation exploits a serendipitous informal extension of the semantics of the diagram in Figure 4 whereby the locality of the nodes of the abstract graph is taken into account. Taken together, such model-making exercises suggest quite different priorities from those normally associated with mathematical modelling. A strict and formal mode of interpretation may suit a Pythagorean or Schenkerian view of music, but does not give the scope for ambiguity that is required in creative musical analysis. It is not only helpful to be able to make mathematical models of many different varieties, but also to explore the semantic relationships between them, and to study them in relation to the discourse surrounding them and the processes that mediate their evolution.

3.2. Model-making from a musical perspective. Though the term "model-making" is not prominently used in musicology, it can be seen as an essential aspect of musical theory and analysis. The range of modelling activities involved is broad, and encompasses far less formal and mathematical approaches than are represented in this paper, but the semantic challenges are similar. The trend in musicological practice has been from an idealized to a pragmatic interpretation of music theory: witness how Caplin [3] restricts his theory of formal functions to *instrumental music of specific composers between 1780 and 1810* [6], and the systematic but *informal* nature of Cooke's attempt to describe the semantics of the language of music [4]. The focus has also shifted to exploratory and culturally mediated activities shaping musical meaning. As Kofi Agawu [1] observes: "… when foundations [of Music] are enshrined in Theory and deployed in something called Theory-based analysis, they do nothing but reproduce themselves. It would be wise, then, in seeking to escape the circularity of theory-based analysis, to return to an issue that exercised several minds as far back as the 1960s, and to consider detaching theory from analysis. Theory is closed, analysis open. A theory-based analysis does not push at frontiers in the way that a non-theory-based one does. Theory is foundational, analysis non-foundational. But analysis is also performance, and its claims are to a different order of knowledge than, say, historical or archival knowledge."

Agawu's agenda calls for model-making that is eclectic, open-ended and embraces informal and experientially mediated semantics. The flexibility of the observational and computational framework required for this purpose is illustrated by Huron's Humdrum Kit [7]. As explained in [2], the models devised for this paper take the form of linked *construals* [17], rather than discrete programs, and give support to match McCarty's dictum [9:27]: "computational models, however finely perfected, are better understood as *temporary states in a process of coming to know* rather than fixed structures of knowledge".

3.3. Morals for the musical analyst and model-builder. The idealized concept of music, inherited from the ancient Greeks, suggests a relationship between mathematics and music that has a transcendent quality. Modern computing technology may, through its apparent emphasis on formal musical syntax and the digitization of sound, promote an abstract view of music as comprehensively captured by a mathematical specification. In reality, mathematical representations of music illustrate a relationship between form and content that has great complexity and subtlety. Computing technology is most significant because it enables us to model this rich and ever evolving relationship more vividly and dynamically than was possible in the past. In our enthusiasm for interpreting music in mathematical terms, we should not lose sight of the pragmatic considerations that enable us to match musical compositions to abstract structures (such as the coincident approximate equality of 3/2 to the power 12 and 2 to the power 7, and the need to discount the creative impulses that inspire composers to deviate from conventional relationships). Nor should we be misled into supposing that model-building with computers commits us to imposing formal mathematical models on music. Mathematics, music and modelling are all activities that derive their creative essence from relationships between formal patterns and experiences that are always fluid and open to negotiation. This flexibility and openness should be reflected in the educational and technological support that is provided for them.

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