Creating Penrose-type Islamic Interlacing Patterns

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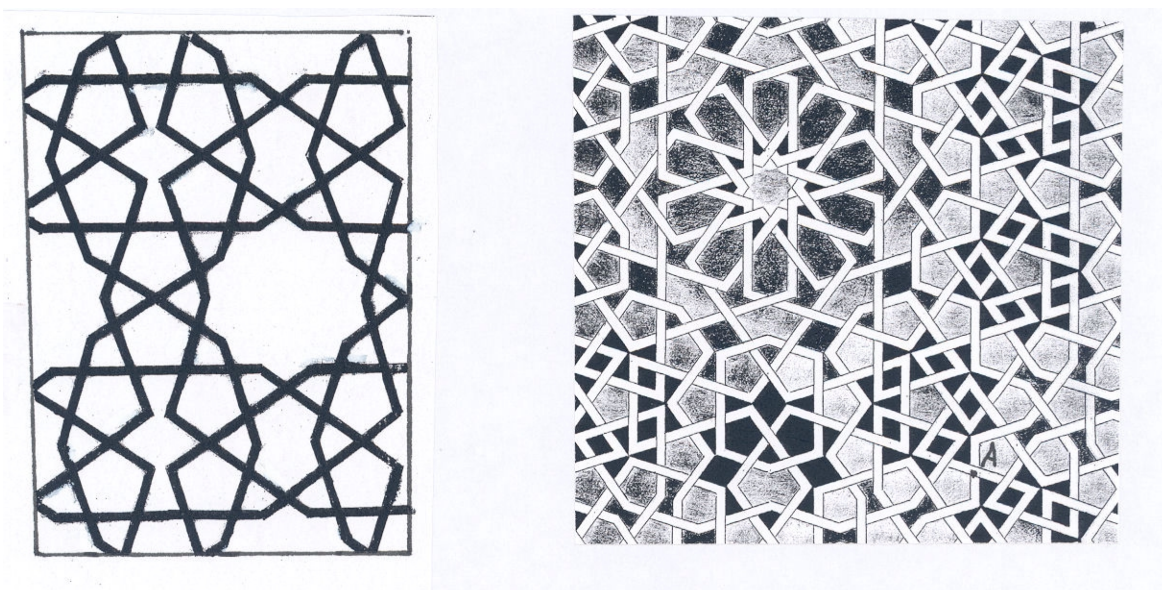
Abstract

Some of the most interesting Islamic interlacing patterns involve ten-pointed stars or ten-petalled rosettes. These motifs have local ten-fold symmetry, yet they are often included as part of a plane periodic pattern, which can have no overall five- or ten-fold symmetries. Instead of using these motifs in periodic patterns, can we incorporate them in patterns based in some way on Penrose tilings (which have many local five-fold symmetries)?



(a) (b)

Figure1. *Interlacing patterns from Afghanistan and Turkey.*

1. Interlacing Patterns and Penrose Tilings

1.1. Interlacing Patterns. Ten-pointed stars (Figure 1a) and ten-petalled rosettes (Figure 1b) occur in Islamic interlacing patterns from many countries. Figure 1b shows part of a very elaborate pattern from the Karatay Medrese at Konya in Turkey**;** horizontal and vertical lines through the centre of the rosette are mirror lines of the pattern if we disregard the under-over interlacing, and the point marked *A* is a 2-fold rotational centre. Although the rosettes and other small motifs in this pattern have local five- or ten-fold symmetry, it is well known that periodic patterns in a plane cannot have any overall five-fold symmetry. One situation in which five-fold symmetry can extend over a complete periodic pattern is on a dodecahedron or on a sphere. This idea does not seem to have been much employed in architecture, but an alcove in the Prince's Room in the Reales Alcázares, Seville, Spain, has a polyhedral domed ceiling containing eight trapezium-shaped planes with 72º and 108º angles, into which ten-petalled rosettes fit neatly.

After giving a brief description of Penrose tilings, we shall show how to create patterns in an Islamic style that, instead of being periodic, share some of the geometrical properties of Penrose tilings.

1.2. Penrose Tilings. Penrose kite-and-dart tilings are aperiodic – they do not repeat regularly. An account of such tilings can be found in [5, Chap. 10], and Figure 2 shows an example**;** we shall simply state some of their basic properties without proof. The kite and dart tiles have side-lengths in the ratio $\tau : 1$ where $\tau = (1 + \sqrt{5})/2$ is the golden ratio, and their angles are multiples of 36º. One way of ensuring aperiodicity is to colour the corners of the tiles alternately black and white**:** the 144º angle (or corner) of each kite, and the 36º angles (or corners) of each dart, are black, and the corners of the tiles meeting at a vertex of the tiling must be either all black or all white. This extra requirement ensures that the vertex-coloured kites and darts can only be put together in an aperiodic manner, as described in [5].

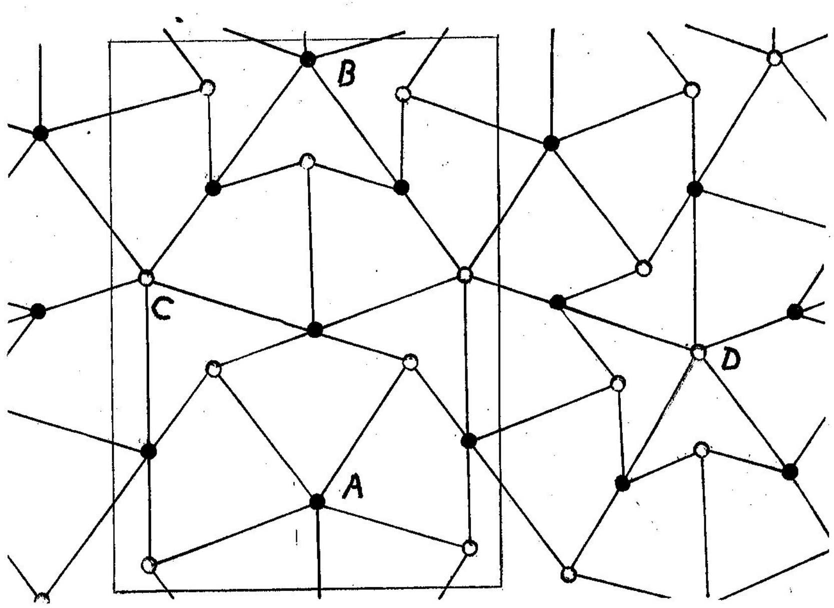


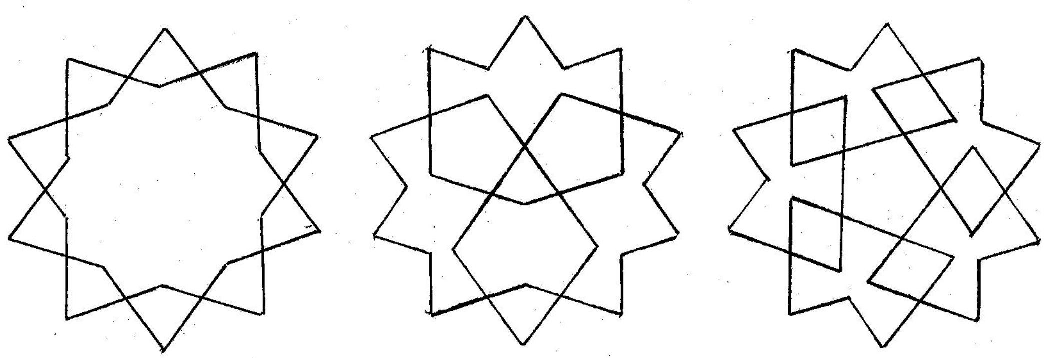
Figure 2. *A Penrose tiling of kites and darts.*

1.3. Penrose-type Patterns. There are two properties that make Penrose tilings of interest to us here**:** although they are aperiodic, they contain arbitrarily large finite regions with five-fold rotational symmetry, and any finite region is repeated infinitely often in the tiling. So, instead of producing periodic Islamic-style patterns associated with the plane symmetry groups, it would be nice to be able to produce Islamic patterns, especially patterns containing star or rosette motifs, that are in some way associated with Penrose tilings. We shall proceed to do this in an exploratory way, and shall wait until the end before attempting to give a definition of the term "Penrose-type pattern". The Karatay pattern in Figure 1b, and a Penrose-type pattern in Karatay style, will form the beginning and the end of our investigation.

2. Creating Penrose-type Patterns

2.1. A Method of Creating Penrose-type Patterns. The obvious way to associate a pattern with a Penrose tiling is to draw part of a pattern on a kite, and part on a dart, in such a way that when a copies of these partial patterns are used on each kite and each dart of a Penrose tiling, (a) the partial patterns fit together neatly whenever two tiles meet edge to edge, and (b) the resulting overall pattern looks Islamic in style. In other words, we use copies of a patterned kite and a patterned dart to create an overall Penrose-type pattern. Satisfying the criteria (a) and (b) is not as easy as it sounds, and (b) involves a subjective judgment.

2.2. Skeletal Islamic Patterns. First we must get used to studying and creating Islamic patterns. When studying the structure and the geometry of an interlacing pattern, it is convenient to use a *skeletal form* of the pattern, in which the interlaced braiding is reduced to a single line**;** the background shapes in the original pattern now become tiles in a tiling, and the braids become the edges of the tiles. The skeletal forms of the star motif from Figure 1a, and the lion head and five-diamond motifs from Figure 1b, are shown in Figure 3, and the rosette from Figure 1b is shown in Figure.4 The term "skeletal pattern" is convenient, but perhaps unfair, since such patterns often occur as ceramic wall patterns in their own right, not just as simplified drawings of braided patterns. (The artist M.C.Escher copied a pattern in the Alhambra which occurs in both skeletal and braided form with only minor variations**;** [1, pp.41, 53].)



Star Lion head Five-diamond

Figure 3. *Skeletal forms of motifs from Figure 1.*

2.3. A Penrose-type Pattern Using a Restricted Set of Tiles. The lion head motif in Figure 3 is made up of shapes of tile that occur frequently in patterns from many sources**:** a regular pentagon with side-length 1 unit, a non-convex 10-gon with sides of length 1, and a non-convex 8-gon that is always combined with two small kites to form a rhombus with side-length 2**;** I shall call these *the four tiles***.** Many interesting patterns can be created using copies of just the four tiles, but such patterns do not seem to occur in Islamic architecture. However, examples in which the four tiles are combined with the star motif (Figure 3) do occur architecturally and in collections of patterns [2, 4, 6]. Figure 1a shows one such example. Whilst collaborating with a colleague in the creation of four-tile patterns – a form of artistic and geometrical relaxation! [8] – I considered the possibility of producing a Penrose-type four-tile pattern**;** to make a start I decided to introduce stars also, and I was then able to devise the patterned kite and dart shown in Figure 5. There is a partial star centred at each corner of the kite and dart, and the half-rhombuses and partial stars on the individual kites and darts fit together to form complete rhombuses and stars. When the patterns of Figure 5 are applied to the portion of Penrose tiling enclosed within the rectangle in Figure 3, the resulting pattern is shown in Figure 6.

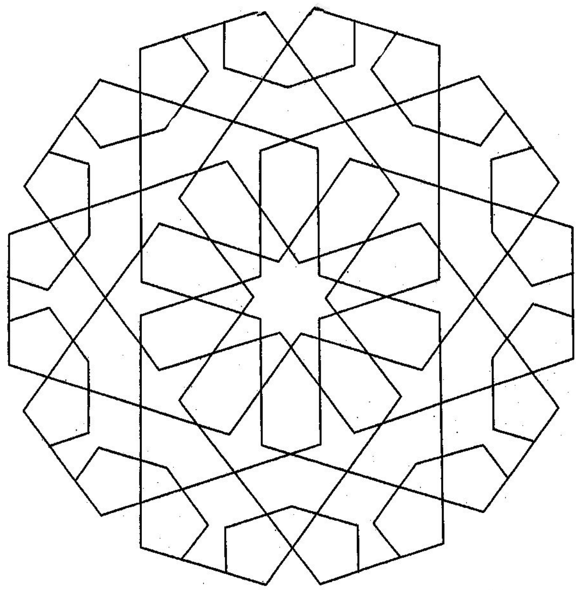


Figure 4. *The skeletal form of the rosette motif.*

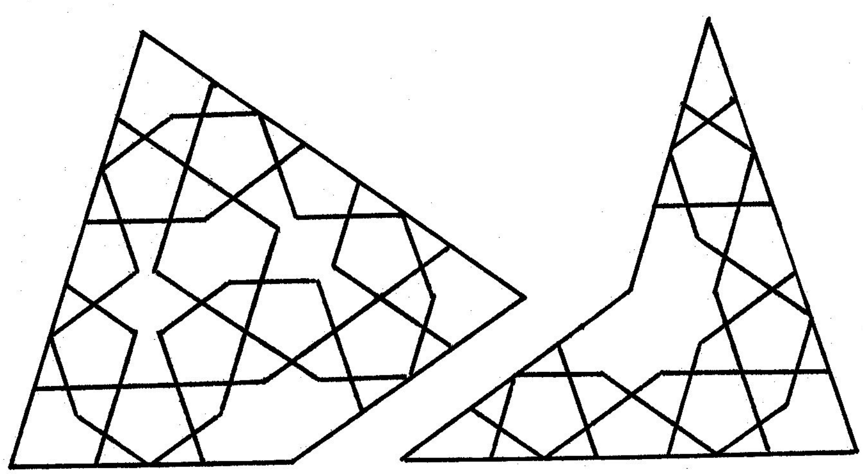


Figure 5. *A patterned kite and dart.*

2.4 A Penrose-type Karatay Pattern. Figure 6 is itself a perfectly good Penrose-type pattern, but at this stage I was still trying to produce a Penrose-type four-tile pattern. My original idea was to replace each star in Figure 6 by a lion head motif, which we can do because the two motifs have the same size and shape, but this process destroys much of the local 5-fold rotational symmetry of the Penrose tiling, so the result can hardly be called Penrose-type. However, having laid out Figure 6 using cardboard tiles, it occurred to me that we can transform this pattern into a pattern having many of the characteristics of the Karatay pattern in Figure 1b. The star marked *A* in the figure is surrounded by a ring of ten pentagons, ten rhombuses and ten 10-gons, then by a ring of twenty pentagons. We notice that the rosette motif in Figure 4 is surrounded in exactly the same way by twenty pentagons, so the star marked *A*, and its immediate surroundings, can be replaced by a rosette motif. At the same time, let us replace the stars closest to star *B* (three of these are shown in the figure, marked *X*, *Y*, *Z*) by lion heads all facing in towards the centre of *B*. The result is shown in Figure 7.

We find that star *B* is now surrounded by twenty pentagons, so it also (with its immediate surroundings) can be replaced by a rosette. Star *C* can be treated in much the same way as star *B*; the steps are not quite the same, but again we finish up with a rosette, as in Figure 8. Finally, since the Karatay pattern contains no stars, we can replace the remaining stars by five-diamond motifs and lion head motifs in some suitable manner.

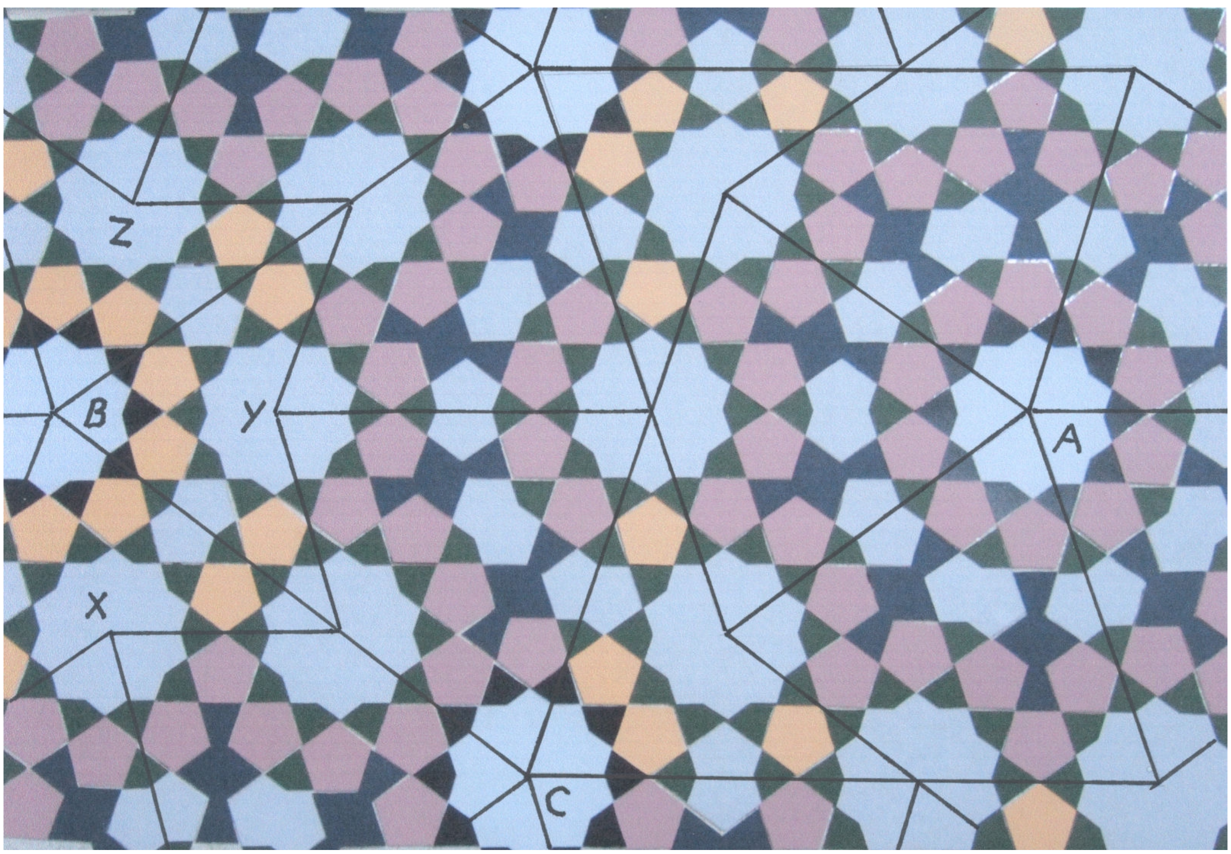


Figure 6. *A basic Penrose-type pattern obtained from Figure 5.*

Note that Figure 6 shows merely a small part of an infinite pattern. The centre of star *A* lies at a point of the underlying tiling (Figure 2) where five kites meet; we subject all stars with this property in the infinite pattern to the same treatment as *A*, replacing them and their surroundings by rosettes. The centre of star *B* lies at a point of the underlying tiling where five darts meet; we subject all stars with this same property to the same treatment as *B*. Star *C* lies at a point of the underlying tiling where four kites and one dart meet; where possible we subject all stars such as *C* to the same treatment as *C*, but this cannot be done if they lie too close to an *A*-star or *B*-star. So, Figures 6 – 8 show small finite portions of infinite patterns;

unfortunately these patterns only begin to exhibit their Penrose quality in a visual manner when we can view a much larger portion than has been possible here.

2.5 Thoughts on a Definition of "Penrose-type". What is meant in the last phrase of the previous paragraph but one by "suitable manner"? What modifications can we make to a pattern whilst still describing it as "Penrose-type"? Here are some thought on the matter, but they fall short of providing a precise definition. Two patches or regions in a pattern or tiling, P and Q say, are *congruent* if there is an isometry (a distance-preserving transformation) transforming the portion of the pattern on P to the portion of the pattern on Q.

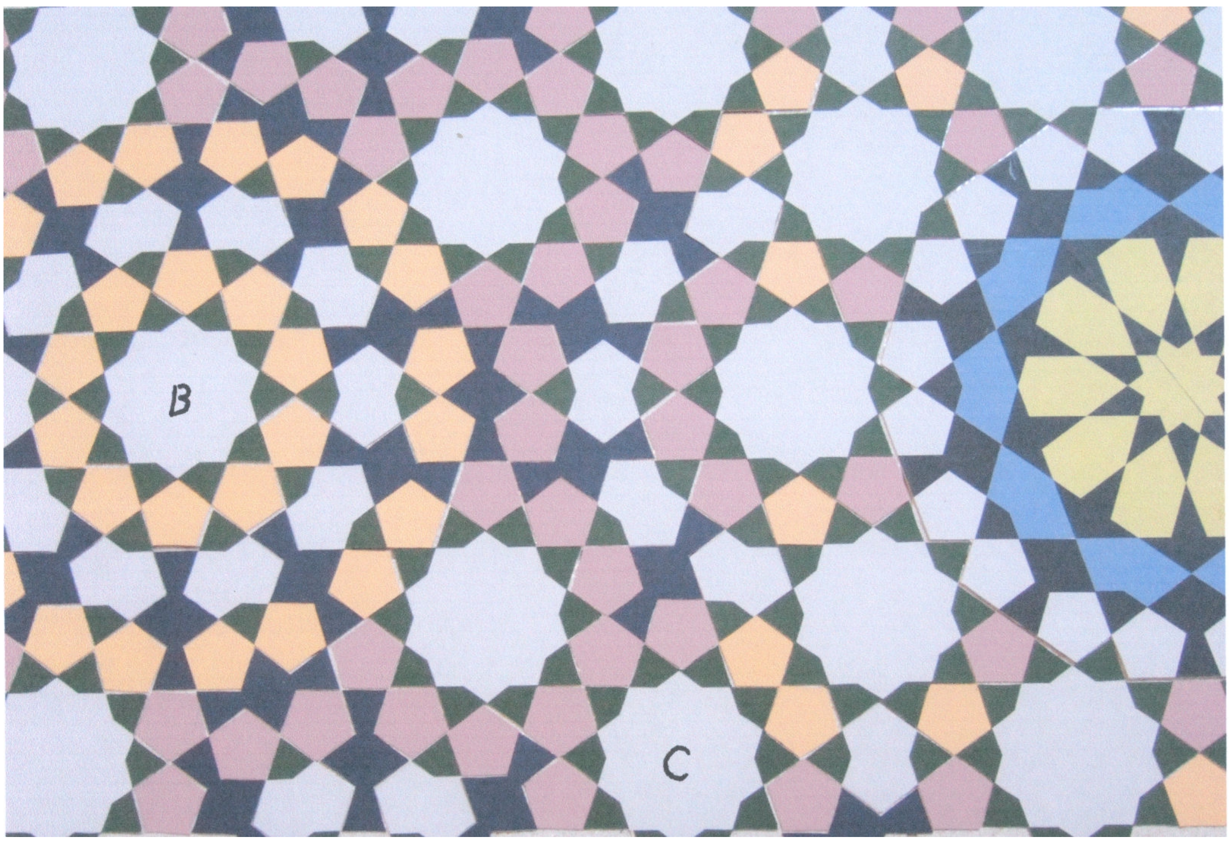


Figure 7. *A modification of the pattern in Figure 6.*

If we start with a Penrose tiling **T**, and replace its kites and darts by patterned kites and darts, as described in 2.1, **T** is transformed into a pattern **T*;** let us agree to say that **T*** is a Penrose-type pattern, or more precisely a *basic* **T**-*type pattern*. To obtain a *modified* **T**-*type pattern*, we replace a patch of tiles P in **T*** by another patch P′ (where the boundaries of P and P′ have the same size and shape) and at the same time we replace *every other patch* Q *congruent to* P by a patch Q′ congruent to P′. (For instance, in 2.4 we replaced all stars of a certain type, with their surrounding tiles, by rosettes.) Any number of such modifications can be carried out.

With this terminology, Figure 6 shows a basic Penrose-type pattern**;** Figures 7 and 8 show modified patterns obtained from Figure 6.

3. Other Penrose-type Patterns

3.1. A Pattern Similar to 2.4. The centre of every star in Figure 5 lies at a vertex of the underlying tiling shown in Figure 3. There are only seven types of vertex in a kite-and-dart tiling [5, p.561] (two vertices are of the same type if they are surrounded in the same way by their adjacent kites and darts), and a brief investigation shows that any star can be subjected to a treatment similar to that given to *C* in 2.4, replacing adjacent stars by lion head motifs and then replacing the star and its surroundings by a rosette. Two stars can be subjected to this treatment simultaneously as long as the distance between their centres is not less than $s + t$, where s and t are the lengths of the short and long edges of the kites and darts.

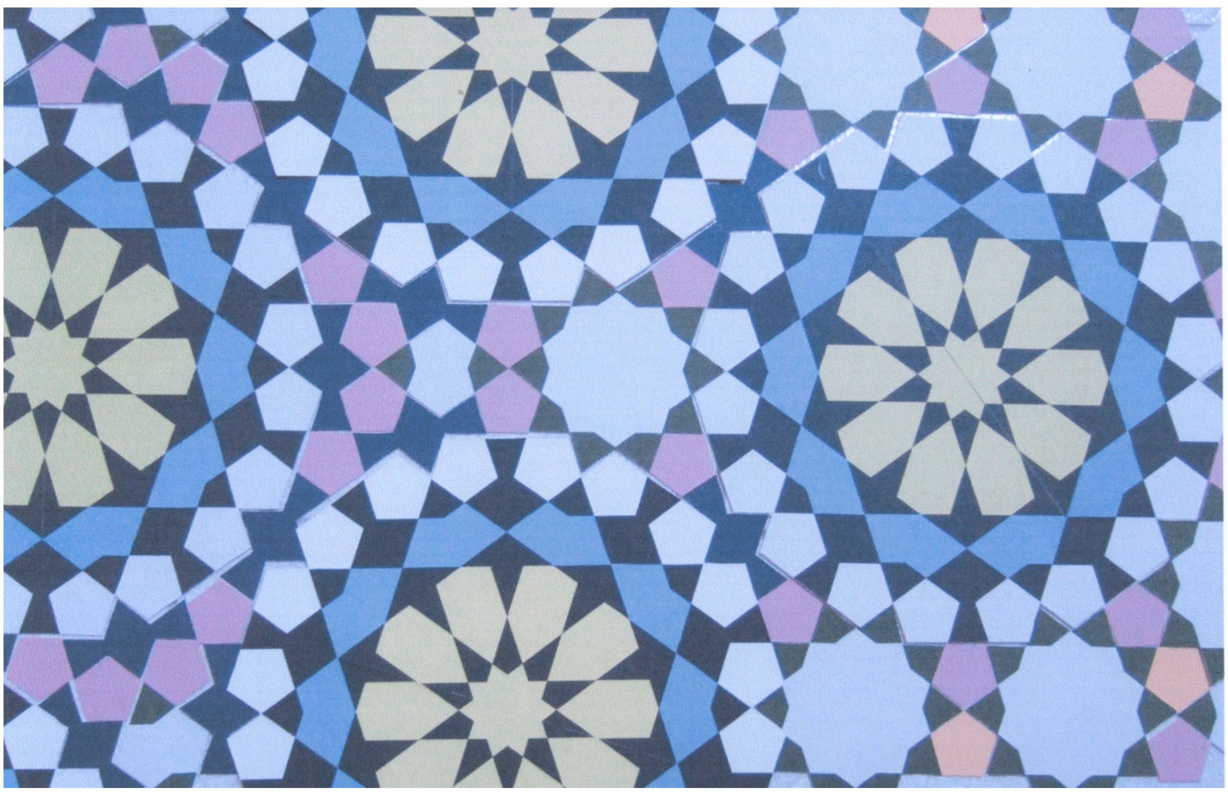


Figure 8. *Further modifications of Figures 6 and 7.*

In 2.4 we replaced all *A*-type and *B*-type stars, and as many *C*-type stars as possible, by rosettes. Another type of vertex in Figure 2 is the one labelled *D*. It is easily verified that if we consider all vertices of types *A*, *B* and *D*, the distance between any two of these vertices is at least $s + t$, so we can replace all the corresponding stars by rosettes to obtain a Penrose-type pattern different from the one in 2.4.

We cannot carry out a similar process on the remaining types of star – there will always be pairs of stars somewhere in the plane that are too close together. But if we are interested merely in creating a pleasing pattern in a small region of the plane, it may be possible to find other stars that can be replaced.

3.2 Other Patterns. It is possible to create Penrose-type patterns using only the four tiles as described in 2.3. As patterns of shapes they are unexciting, but suitable colouring brings them to life. J.-M. Castéra [3, p.287] describes a different method of creating Penrose-type patterns based on Penrose rhombs. There is not space here to describe and illustrate these ideas, but it may be possible to include them in my presentation. We conclude with another pattern, which seems more elaborate because we can see more of it in Figure 9. The centre of the rosette in the middle of the top edge is a 5-fold rotational centre of the complete pattern

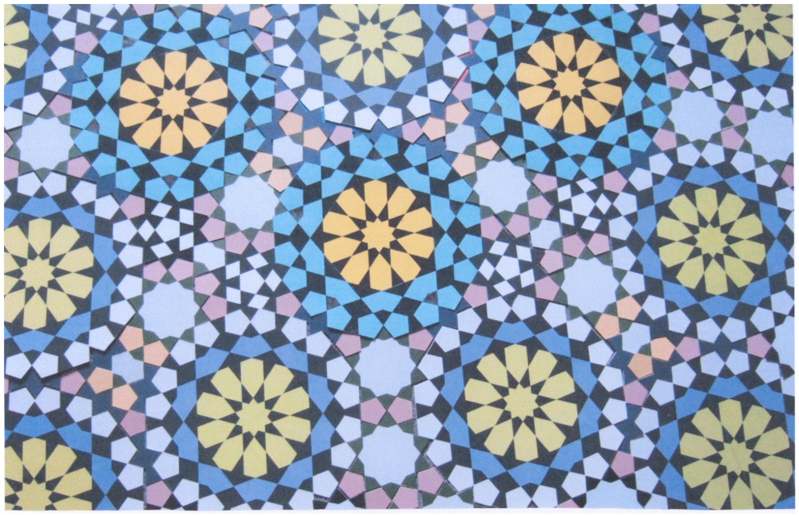


Figure 9. *Another Penrose-type pattern obtained by a similar process.*

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