Asymmetry vs. Symmetry in a New Class of Space-Filling Curves

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Abstract

A novel Peano curve construction technique shows how the self-referential interplay between symmetry and asymmetry based on the translation, rotation, scaling, and mirroring of a single angled line segment that traverses a square evinces rich visual beauty and optical intrigue.

Consider the patterns, or geometric motifs, labeled **C**, **D,** and **E** in Figure 1. Each is a sequence of connected line segments that differs slightly from its neighbor:

C **D** **E**

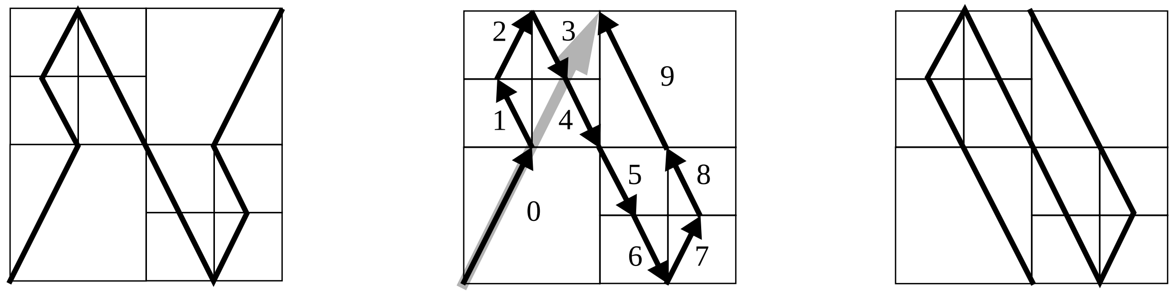
Figure 1: *Three patterns each differing from its neighbor by one line segment*

Before continuing, take a moment to think about the answers to the following aesthetic questions:

Which pattern, **C, D,** or **E**, is more *symmetric*?
And, more subjectively, which pattern is more *elegant*, *pleasing to the eye*, and/or *beautiful*?

Aesthetic questions and answers are of course subject to personal whim, but they still serve to illuminate the makings of interesting mathematical art. So we first analyze the geometry underlying patterns **C**, **D**, and **E** to show that the foregoing questions have an unexpected answer.

As Figure 2 shows, each of these three motifs is embedded within a square that is divided into a simple, rotationally symmetric arrangement of ten smaller squares, in two sizes, that tile (cover without gap or overlap) or dissect the original. All three patterns are the same length, and each traverses through the same set of smaller square tiles in the same order. Each contains line segments that are translationally parallel to other segments, which is a visually pleasing form of internal reference. Patterns **C** and **E** are rotationally symmetric 180° about the center, whereas **D** is not. None exhibits either vertical or horizontal bilateral symmetry. All three patterns exhibit an identical internal duplication: the upper-left quadrant (containing four smaller square tiles) of each is congruent with the lower right quadrant, rotated 180° about the center. Again, that internal central point reflection makes all three patterns visually interesting by virtue of the internal correlation and attendant partial rotational symmetry. Yet pattern **D** seems symmetrically awry, as if it were a poor compromise between **C** and **E**, both of which could be used in a simple, translationally repetitive frieze.



C **D** **E**

Figure 2: *Each pattern traverses the same 10 smaller squares that cover the original*

While patterns **C** and **E** exhibit more symmetry, thereby perhaps winning the aesthetic beauty contest, a closer look at **D** shows that it has a deeply elegant, geometric property that its neighbors are missing.

Pattern **D** is a path that traverses its enclosing square by travelling from a corner to the center of an opposite side (gray arrow, above). But the internal smaller square edges divide the pattern into segments, several of which are co-linear. Thus, if one takes **D** as composed not of six but of ten connected, directed line segments, each of these ten segments traverses—while maintaining piecewise continuity—exactly one of the ten sub-squares *in essentially the same geometric manner as **D** traverses its own square*. Each segment connects its smaller square's corner to the center of an opposite side, in either a right- or left-handed manner, and in either a forward or backward direction, and in various of four possible rotated orientations.

Thus, while all three patterns are essentially tile designs, pattern **D** is a spatially recursive "detour" that is composed of parts that each accomplish the same goal as the whole does, but at two smaller scales. Because each smaller square can itself be similarly subdivided, each leg of the **D**-tour can be replaced—after a simple linear transformation—by a smaller half- or quarter-size **D**-tour, yielding a **D**-**D**-tour (denoted $\mathbf{D}^2$), all without affecting the continuity of the overall tour The limit of the convergent sequence of $\mathbf{D}^n$-tours will be a space-filling Peano curve [1][2] that passes every point in the original bounding square region. The classic Hilbert and Peano Curves can be analysed using base 4 or 9 number systems [1], but in this new construction, simply label the appropriately ordered ten subsquares with the digits 0...9 (Figure 2, center). Any infinite decimal expansion of a point in the interval [0,1) maps—under the appropriate geometric ordering of subsquares and iterated linear transformations, one per digit, each scaling by at most 1/2—onto a point in the square.

Like Peano's original 3x3 (analytic) construction, each finite stage of the composition increases in length by a factor of 3. Unlike the original Peano Curve, here finite stages contain copies of previous, less-detailed stages at different sizes, more akin to Mandelbrot's "Snowflake Sweep" construction [3]. The multitude of different scales visually draws out the recursive quartering of the square and all its subsquares. Hence, I enjoy calling this space-filling curve construction the "Peano Quartet" (Figures 3 and 4):

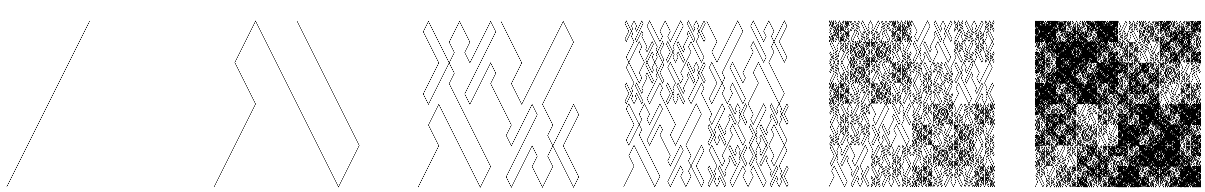


Figure 3: *Stages 0 through 5 of Peano Quartet space-filling curve construction*

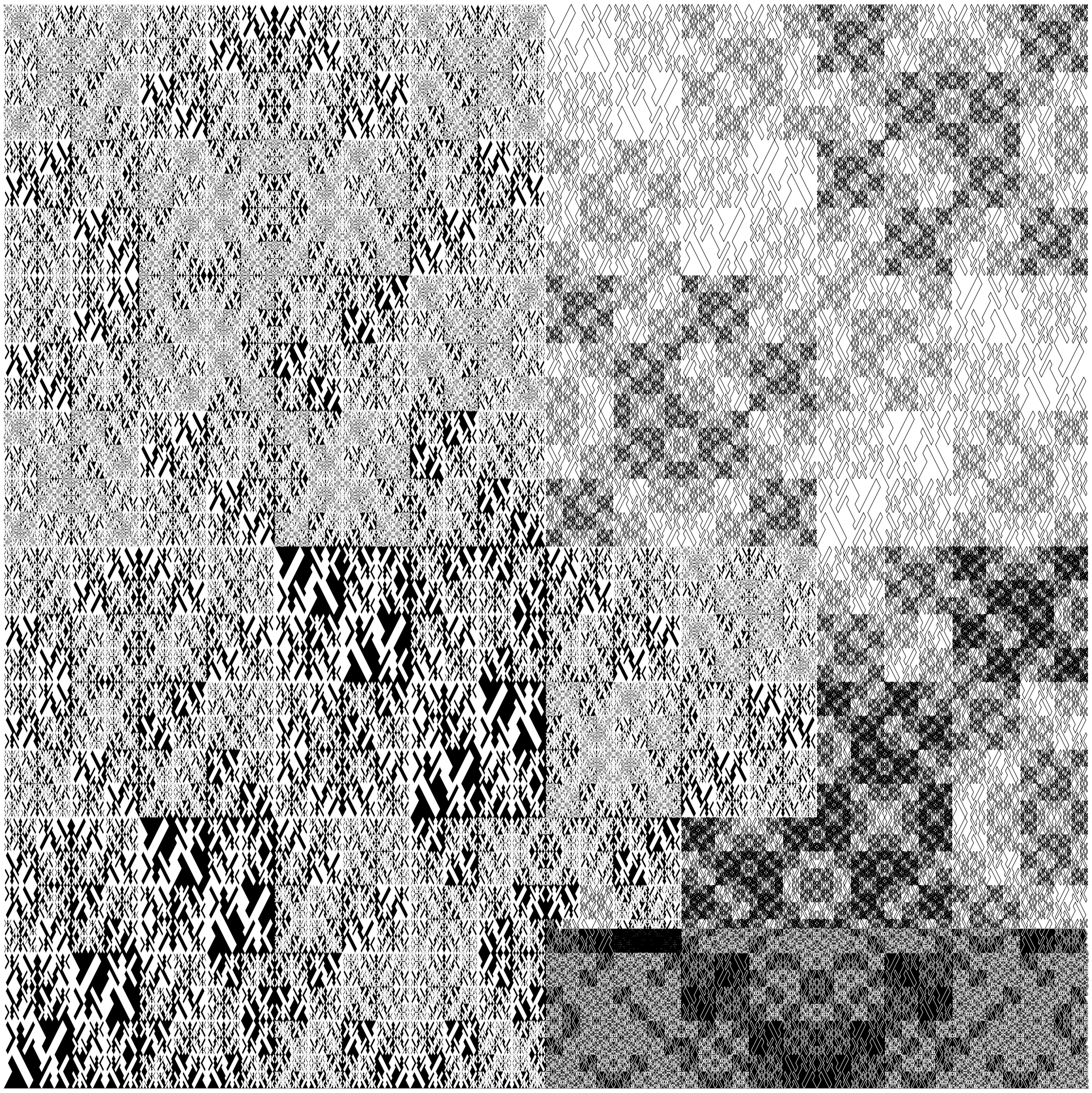


Figure 4: *Stage 6 of the Peano Quartet contains 10^6 line segments*

The original Peano Curve's highly regular approximation paths are self-contacting at every other lattice point [3]. But in the Quartet construction, finite approximations are nearly everywhere self-avoiding ($\mathbf{D}^2$ with 100 segments has only 2 points of self-contact, and $\mathbf{D}^3$ with 1000 segments has only 29). This is due primarily to the fact that, other than the lower left corner once, no finite $\mathbf{D}^n$-tour can ever contact the logical right or left side of its embedding square. Uniform, square-filling, edge-replacement Peano curves, such as McKenna's E-Curve and its relatives [2][4], cannot be completely self-avoiding (at finite stages) below order 5. In the limit, though, all space-filling curves are self-contacting, surjective mappings [1].

Because all finite $\mathbf{D}^n$-tours inherit $\mathbf{D}$'s essential asymmetry, but still have internal parts that are congruent, each finite stage displays a wonderful visual tension between high-level symmetry and low-level asymmetry. When thin lines with finite thickness illustrate the path, the parts drawn at the smallest scales will be darker. This results in a rich binary quilt (Figure 4) composed of repetitions of the initial diagonal line. The upper-left quadrant is congruent under 180° rotation about the center with the lower-right quadrant. The remaining two quadrants, however, are not congruent.

Self-avoidance of an approximation path is both visually important and combinatorially constraining. Each finite Peano Quartet approximation ($\mathbf{D}^n$) is a piecewise linear, one-dimensional curve that—regardless of self-contact at some subsquare corner—divides its world into two parts. When one colors one side of the path black and the other side white, in addition to handedness one can also have visually positive or negative subsquares that correspond to the direction the curve is travelling across that square. The nearly complete (finite) self-avoidance means that large white and black regions of the traversed square are topologically connected at more than just single points of self-contact. This hides the underlying geometry and leads the eye/brain to integrate larger scale structures and imbue connected areas with visual form and meaning. This is in striking contrast to the classic Peano Curve, whose approximations, when similarly colored, create nothing more than a uniform, diagonal checkerboard pattern [3]. But here, different sizes fool the eye into seeing distances and depth, foreground and background take on the same shapes, and an unexpected and intriguing visual property—an optical illusion—appears: the horizontal binary division lines are parallel, but rather unsettlingly do not look so (Figure 5):

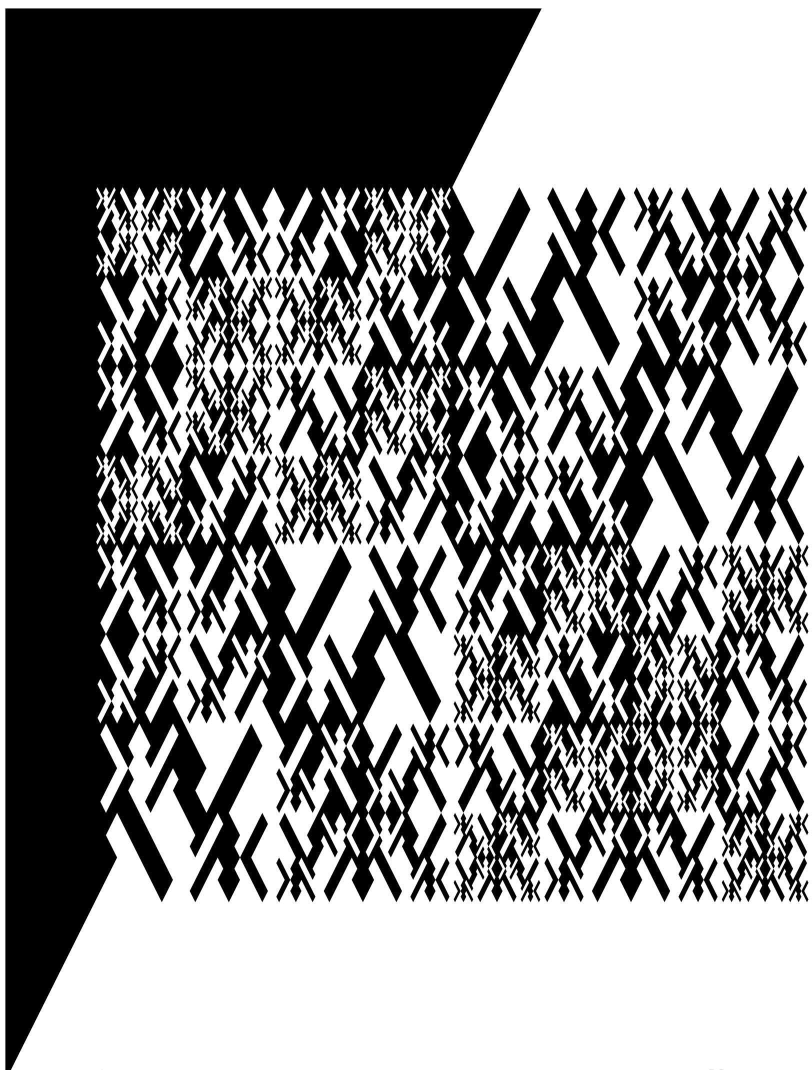


Figure 5: *“Peano Quartet”, 2005 (36 in. x 40 in., giclèe print)*

The question arises as to the uniqueness of the construction. The combinatoric space is highly constrained, especially at the four corner squares where, regardless of their sizes, path connectivity from corner to top center requires that all four corner subsquares each have one of only two possible directed line segments traversing them (Figure 6). Pattern **D** is the only possible ten-segment, self-avoiding traversal from lower-

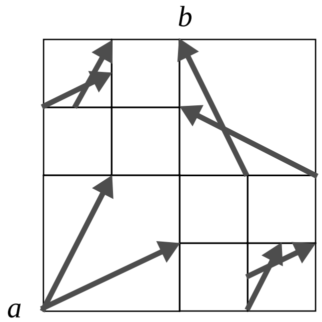


Figure 6:
Corner squares admit only two possible segment choices as part of any self-avoiding directed traversal from a to b

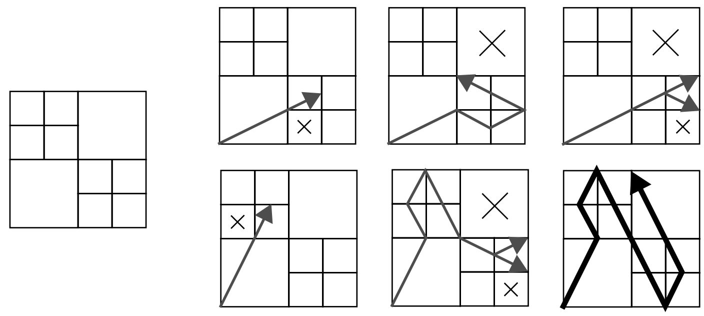


Figure 7:
Pattern ***D*** *is unique for given dissection into 10 subsquares. Squares for which it is impossible to continue building a solution are marked with an* X.

left corner to the center of the top side of the Peano Quartet's underlying dissection (Figure 7). Any other choices of path extension lead to contradictory situations. Because of the asymmetry of the corner-to-center traversal goal, even a simple rotation or reflection of the square's dissection arrangement can change the solvability of the problem. For instance, when the Quartet's underlying dissection is rotated 90°, there are no solutions (Figure 8).

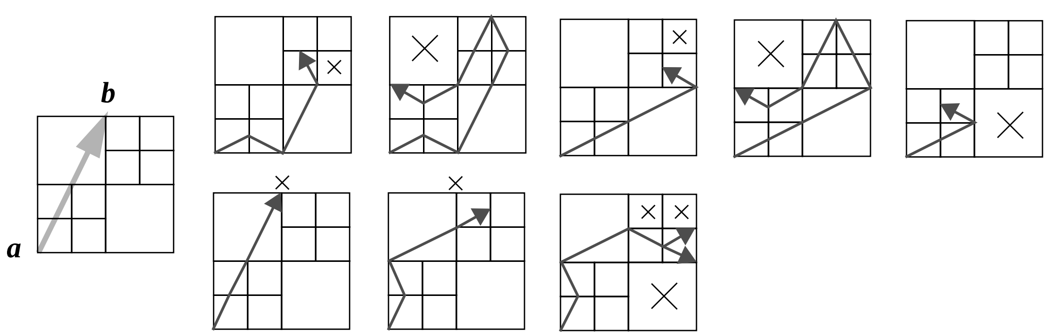


Figure 8: *No solutions from a to b when Quartet's dissection is rotated 90°. An* X *at point b means the path has prematurely arrived, or can never arrive given the current partial path.*

Among the 16 possible dissections (counting rotations and reflections) of a square into ten subsquares, two a quarter size and eight a sixteenth size (Figure 9), hand enumeration shows that there exist 10 space-filling curve generators that are locally self-avoiding (Figure 10). None of these generators is symmetric, and local self-avoidance is necessary, but not sufficient, for global self-avoidance of higher level stages. Nonetheless, many of them generate curve approximations that are largely without self-contact, leading to larger visual features with which the eye can play. Interestingly, one solution is the same shape as pattern **D**.

The recursive tiling patterns that these asymmetric generators self-referentially specify are more successful aesthetically when the underlying dissection has its own symmetries that more detailed stages inherit.

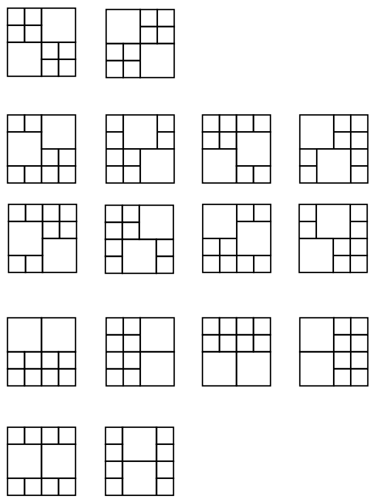


Figure 9: *16 dissections, counting rotations and reflections*

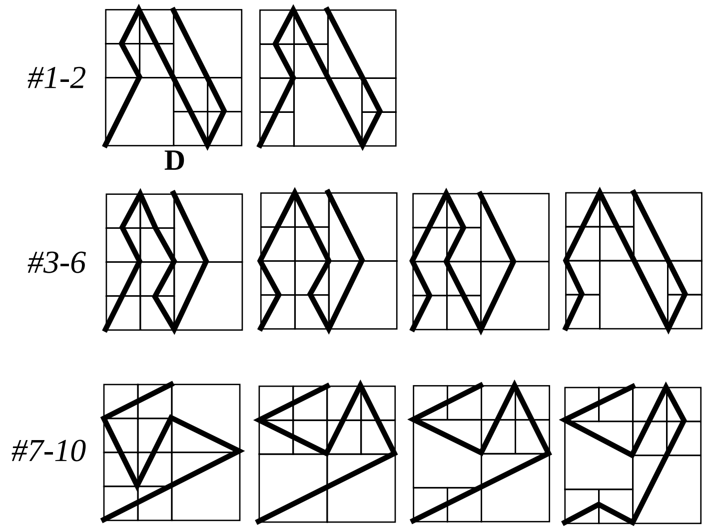


Figure 10: *10 space-filling curve generators. #7 is the basis for the print below*

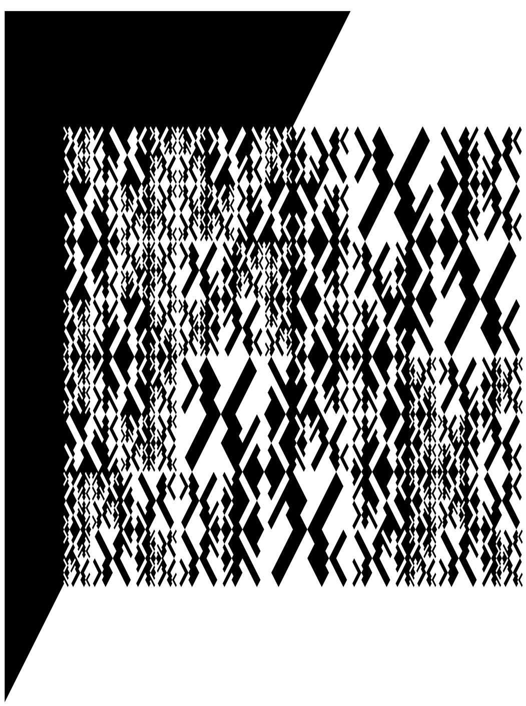


Figure 11: *The Peano Quartet's fraternal twin has fewer symmetries*

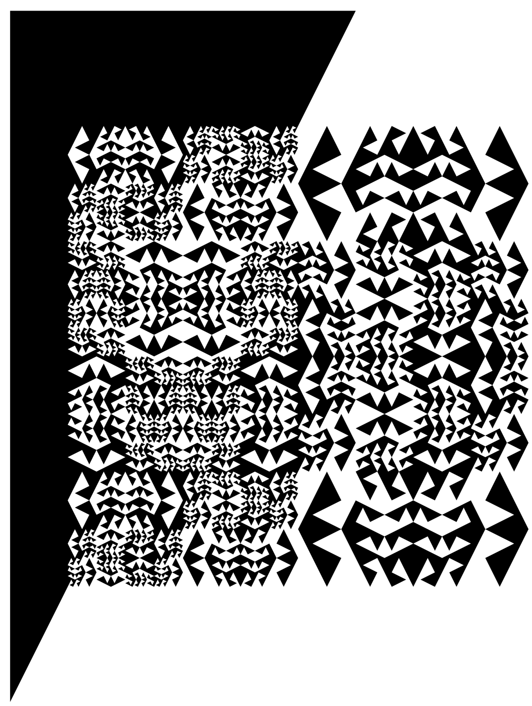


Figure 12: *"Platonic Dance" (giclée, 2006)*

Compare the more chaotic feeling of Figure 11 to Figure 12 ("Platonic Dance"), where the latter exhibits strong bilateral symmetries in many different scales and orientations, because the originating dissection does. Figure 13 ("Blade Rainer") is based on stage 4 of generator #4 from the above list, altered at the final drawing stage to be completely self-avoiding. Figure 14 is a self-avoiding variation based on generator #8. Both are wildly different in visual and even emotional impact and feel from each other (and from either of the above two), by virtue of how our mind and eye integrate and recognize connected forms.

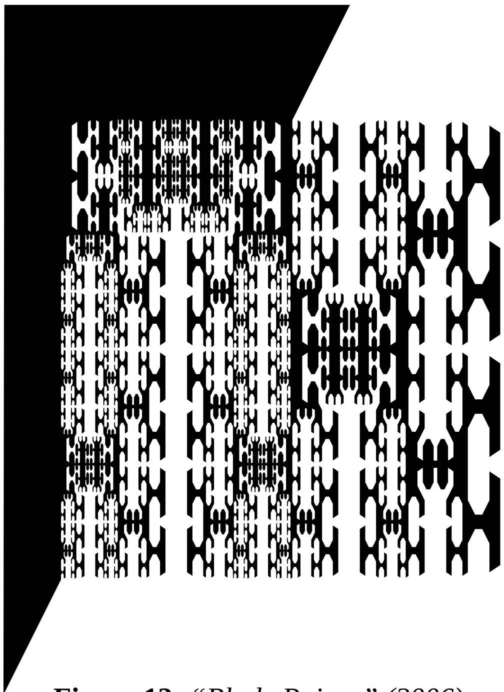


Figure 13*: "Blade Rainer" (2006)*

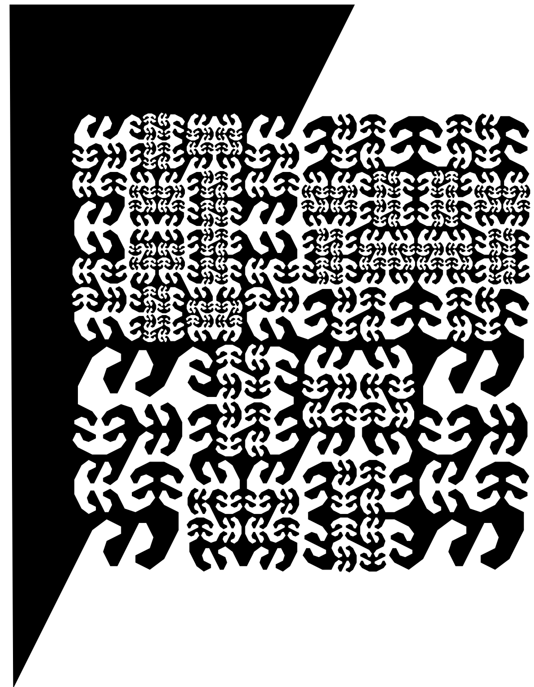


Figure 14*: "What do you see?" (2006)*

If one restricts generator patterns to be self-avoiding and only traverse equal-sized subsquares, there are several locally, but not globally, self-avoiding generators on the order 4 (4 x 4) subdivision). For example, patterns **A** and **B** each generate a more uniformly self-contacting, space-filling curve that, when colored as a tiling at, e.g., stage 3, is visually reminiscent of the textiles of some South American cultures (Figure 15).

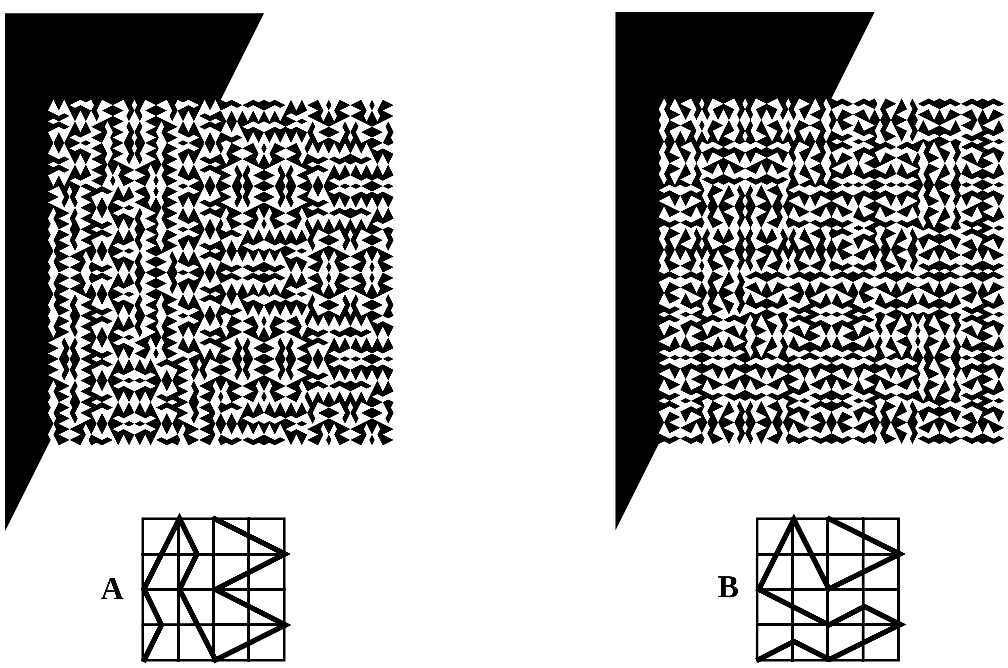


Figure 15: *Stage 3 of two space-filling generators* ***A*** *and* ***B****, within 4 x 4 tile designs*

Neither of these is as visually rich as the more exuberant Peano Quartet and its brethren, because the hierarchical nature of the $\mathbf{A}^n$ and $\mathbf{B}^n$ stages is hidden by the homogeneity of the subsquare scales at any given construction stage.

Whenever non-mathematical visitors view the large framed copy of the Peano Quartet (Figure 5) that I have on my wall, they are visually fascinated, and invariably think I'm making choices, following my muse. I tell them that the Peano Quartet is a composition, but that it is a mathematical one (all puns intended). Yet the crystalline, recursively raucous, connected collection of symmetries and asymmetries seems deeply important to treating these compositions as purely aesthetic objects with inherent visual intrigue and meaning to the viewer. Whether they are more platonically than artistically beautiful is an interesting and ancient question, because art, like math, is created/discovered through a person's choices and exploration within some constrained, combinatoric space. Which is why, when I printed one of my first copies of the Peano Quartet, I labeled it as both "A/P" (Artist's Proof) and "D/P" (Discoverer's Proof). The viewer can then interpret it mathemæsthetically according to their own philosophical slant.

Conclusion

The Peano Quartet's generator pattern **D** is one of 10 possible space-filling curve generator patterns that traverse a dissection of a 4 x 4 square into any of 16 spatial arrangements of two 2 x 2 subsquares and eight 1 x 1 subsquares. Each generator constitutes the asymmetric, self-referential, geometric code that builds a rich tapestry of threaded subsquares that cover an original square, in the limit as a space-filling curve. Finite approximations of these recursive constructions create combined curve and tiling patterns that are much more intriguing and dynamic to the eye than the more commonly known space-filling constructions based on line segments all of equal length.

Beauty in both math and art evinces itself more deeply and interestingly when there's a tension between symmetry and asymmetry. Pattern **D** and its fellow generators are just as symmetrically elegant in their own hierarchical scaling way as patterns **C** and **E** are rotationally. But recursive structure combined with heterogeneous sizes makes **D** and its fellow detours especially intriguing, both visually and combinatorially, more so than the merely pleasant patterns **C** and **E**, whose symmetry is self-evident but only skin-deep. The aesthetic value of these square-traversal patterns also trumps those created by **A** and **B**, where underlying self-referential, hierarchical structures are washed out by the drone of their "one-note" scales.

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