Transformations of Vertices, Edges and Faces to Derive Polyhedra

Robert McDermott
Center for High Performance Computing
University of Utah
Salt Lake City, Utah, 84112, USA
Email: mcdermott@chpc.utah.edu

Abstract

Three geometric transformations produced a large number of polyhedra, each originating from an initial polyhedron. In the first transformation, vertices were slid along edges and across faces producing ***nested*** polyhedra. A second transformation produced ***dual*** polyhedra, whereby edges of the initial polyhedron were rotated and scaled and the end points of these edges derived the dual polyhedra. In a third transformation, faces of an initial polyhedron were rotated and scaled producing ***snub*** polyhedra. The vertices of these rotated and scaled faces were used to derive other polyhedra.

This geometric approach which derives new vertices from previous vertices, edges and faces, produced precise results. A CD-ROM accompanying this paper contains three animations and data for all the derived polyhedra. This CD-ROM can be obtained by sending me email.

1. Introduction

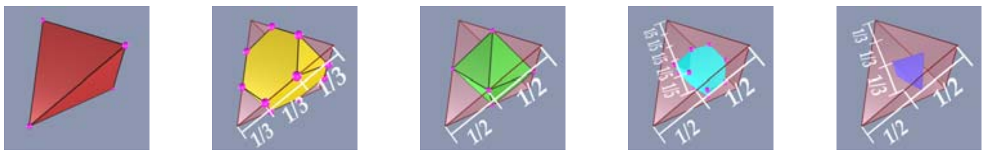
While I was working at NYIT Computer Graphics Laboratory from 1980 to 1990, Haresh Lalvani approached me to produce precise data for an icosahedron and a dodecahedron. He was interested in constructing a quasi-crystal structure that contained more than 600 instances of these two polyhedra. Being an architect, he imagined exact representations. These polyhedra needed to be precise in their spatial coordinates when so many polyhedra were assembled. Any error would have a tendancy to accumulate significantly and prevent the structure from registering all the polyhedra into their respective locations.

I searched for such spatially accurate polyhedra in [1], and [2]. Unable to find precise vertex coordinates, I undertook to develop such polyhedra and subsequently developed other polyhedra: Platonic, Archimedean, Prisms, Anti-Prisms and their Duals. I began by positioning the vertices of a tetrahedron at the corners of a unit cube. These vertices were represented by zeros and ones, making them precise.

2. Vertices are Translated Along Edges & Across Faces to Derive *Nested* Polyhedra

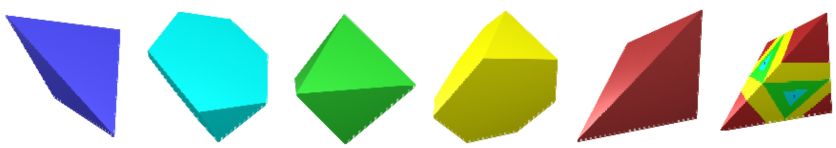
From the vertices of the tetrahedron in **Figure 1a**, a ***nested*** set of polyhedra were derived. I could derive a truncated tetrahedron in **Figure 1b** by translating vertices 1/3 of the length along the edges of the initial tetrahedron. This translation is defined computationally by taking the two end points of the edge, adding the coordinates together, and dividing by the ratio of the proportion along the edge. This very simple computation produced very little error.

By continuing to translate vertices to the halfway point along the initial edges, I derived an octahedron, **Figure 1c**. Now I turned the direction of translation from along the edges to across the face to derive other polyhedra. When I translated the vertices 1/5 the distance from the mid-edge points to the opposite vertices, another truncated tetrahedron was produced **Figure 1d**. When the translated vertices reached the mid-faces or 1/3 the distance from the mid-edge points to the opposite vertices, a dual tetrahedron was derived, **Figure 1e**. In **Figure 2** these five polyhedra are displayed individually and then ***nested*** together.



a **b** **c** **d** **e**

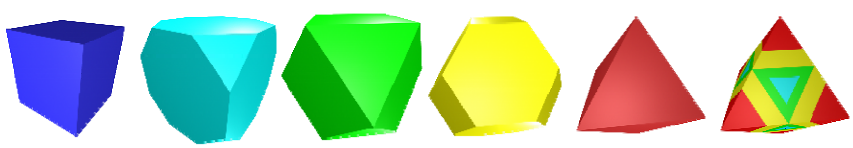
Figure 1: Tetrahedron (**a** & **e**), truncated tetrahedron (**b** & **d**), octahedron (**c**).



a **b** **c** **d** **e** **f**

Figure 2: Tetrahedron (**a** & **e**), truncated tetrahedron (**b** & **d**), octahedron (**c**), & ***nested*** (**f**).

I then applied this ***nesting*** concept to an octahedron. I translated vertices to the 1/3 point of edges to derive the vertices of a truncated octahedron. When vertices were at the 1/2 point of the edge a cuboctahedron was derived. Again, I translated vertices across the faces. When vertices are at 1/5 that distance a truncated cube was derived. When the vertices were at the mid-faces a cube was derived. These five polyhedra can be viewed individually or ***nested*** in **Figure 3**.



a **b** **c** **d** **e** **f**

Figure 3: Cube (**a**), truncated cube (**b**), cuboctahedron (**c**), truncated octahedron (**d**), octahedron (**e**) & ***nested*** (**f**).

In addition, the vertices of an icosahedron were translated along its edges and faces to derive a truncated icosahedron, an icosidodecahedron, a truncated dodecahedron and a dodecahedron. When I initially performed this work, I utilized Coxeter's [1] formulation for the coordinate values of vertices to generate an icosahedron. Five polyhedra and a ***nesting*** of these polyhedra derived from an icosahedron are illustrated in **Figure 4**.

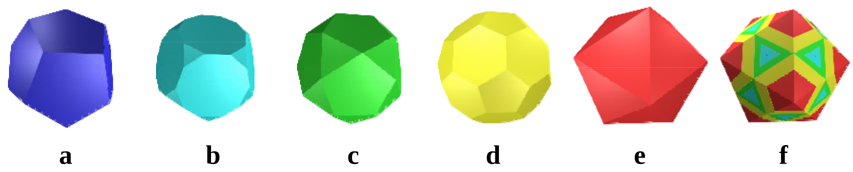


Figure 4: Dodecahedron (**a**), truncated dodecahedron (**b**), icosidodecahedron (**c**), truncated icosahedron (**d**), icosahedron (**e**) & ***nested*** (**f**).

3. Platonic & Some Archimedean Polyhedra

The Platonic and Archimedean polyhedra produced above were the only polyhedra able to be derived from such a simple translation of vertices along edges to their mid-point and across faces to their mid-point.

4. Translating Vertices Along Edges and Across Faces to Square Faces

A slightly new concept was used to translate vertices along edges, and then translate vertices across faces to form square faces. This concept was applied to a cuboctahedron and an icosidodecahedron to derive truncated and rhombic polyhedra. The vertices were translated 1/3 of the distance along edges to form a polyhedron that was topologically equivalent to a truncated cuboctahedron. However, the truncated faces formed were rectangles, not squares. **Figure 5**(**a-d**) are such derived polyhedra with rectangular faces. These rectangles were the result of each vertex having two triangles and two squares. The new edges derived from the triangles differ in length from the new edges derived from the squares; hence the rectangles.

To transform these rectangles into squares, I used the mid-point of each of these rectangles and translated vertices of rectangles toward these mid-points to form squares. **Figure 5**(**e-h**) are such polyhedra with square faces. Consequently, a truncated cuboctahedron **Figure 5e** and a rhombicuboctahedron **Figure 5f** were derived from a cuboctahedron, and a truncated icosidodecahedron **Figure 5g** and a rhombicosidodecahedron **Figure 5h** from an icosidodecahedron.

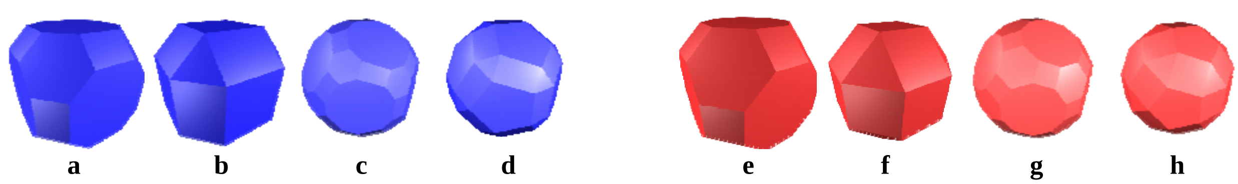


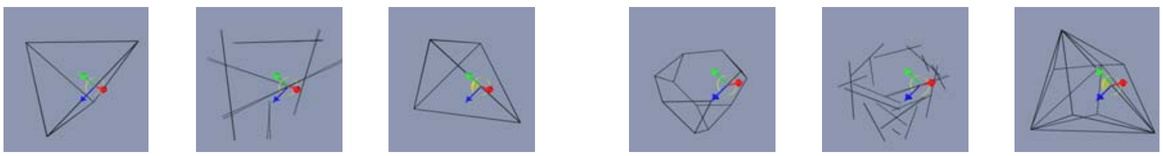
Figure 5: Rectangle faces (**a, b**, **c**, **d**), square faces (**e, f**, **g**, **h**).

5. Rotating and Scaling of Edges to Define *Dual* Polyhedra

I next investigated the transforming of edges to see what would result. The axes of rotation were defined from the center of a polyhedron orthogonal to points on edges. I rotated each edge about each of these axes. The angle of rotation was 90 degrees, so that the rotated edges were orthogonal to the initial edges. For example, I rotated the six edges of a tetrahedron in **Figure 6a**, by 90 degrees to form another tetrahedron in **Figure 6c**. This second tetrahedron was the dual of the initial tetrahedron. For the tetrahedron there is no scaling of the rotated edges. The end points of the rotated edges intersect exactly at the vertices of the dual tetrahedron.

A dual relationship for two polyhedra is an interchange between faces and vertices. That is, faces of an initial polyhedron become vertices in the dual polyhedron, and vertices in the initial polyhedron become faces of the dual polyhedron. When an initial polyhedron and dual are displayed simultaneously it is referred to as a polyhedron ***compound***.

Figure 6 (**a-c**) are rotating edges without scaling for tetrahedron. **Figure 6** (**d-f**) are edges both rotated and scaled for a truncated tetrahedron and a triakis tetrahedron. **Figure 7** (**a-f**) are two sets of an initial polyhedra (**a**&**d)**, polyhedra compounds (**b**&**e**) and polyhedra duals (**c**&**f**).



a **b** **c** **d** **e** **f**

Figure 6: Edges of a tetrahedron (**a-c**), a truncated tetrahedron, & a triakis tetrahedron (**d-f**).

a **b** **c** **d** **e** **f**

Figure 7: Tetrahedron (**a** & **c**), composite (**b**), truncated tetrahedron (**d**), composite (**e**) & triakis tetrahedron (**f**).

6. Dual Polyhedra for Platonic, Archimedean, Prisms and Anti-Prisms

The tetrahedron is a self dual. Therefore, scaling was not required to derive its dual when using the concept of rotating edges for ***duals***. The rotating and scaling edges for ***duals*** is a generalized concept for deriving dual polyhedra. The scaling of the edges is relative to the point where the axis of rotation for the edge and that edge intersect. In most cases there are two scale factors, one for each half edge. A precise scaling value was derived by computing the distance between the end points of edges that were rotated by 90 degrees about their axis of rotation. Each half edge with a like configuration of adjoining polygons had the same scale factor. Once the scale factors were applied for all half edges, the rotated edges intersected precisely in a single vertex of the dual polyhedron.

This concept of simultaneous rotation by 90 degrees and scaling of half edges was applied to Archimedean, Prism and Anti-Prism to define all of their dual polyhedra.

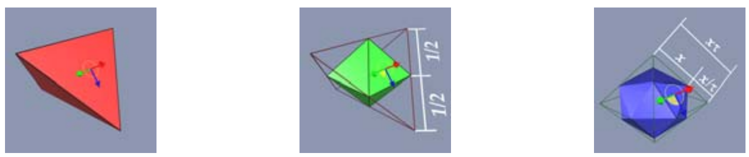
7. Rotating and Scaling of Faces to Derive *Snub* Polyhedra

After translating vertices and rotating and scaling of edges, simultaneously rotating and scaling faces was investigated. The axis of rotation for transforming faces was a line from the center of the polyhedron to a point on a face where the line was orthogonal to the face. Scaling of the faces was performed with respect to the point where the axis of rotation intersected the face. There was only one scale factor used for all faces that were transformed.

The four faces of a tetrahedron, as in **Figure 8a**, were rotated by 60 degrees and scaled by ½ to derive the six vertices of an octahedron, as in **Figure 8b**. This was a second method of deriving an octahedron from a tetrahedron. Note that the tetrahedron has vertices with three fold symmetry and an octahedron has vertices with four fold symmetry.

8. A Snub Tetrahedron is An Icosahedron

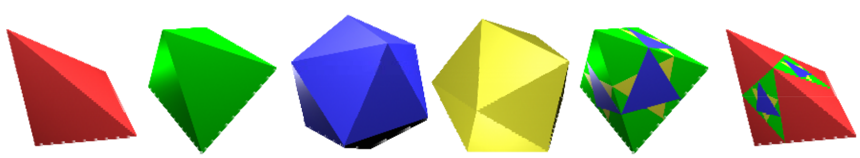
Starting with a tetrahedron as in **Figure 8a**, I used the four triangle faces, consisting of twelve vertices, rotated them by 60 degrees, and scaled them by ½. These vertices formed an octahedron, as in **Figure 8b**. When the vertices of these newly transformed triangles were rotated and scaled further, they appeared to move along the edge of the triangle of the octahedron, to a point, where the ratio of the two pieces of this edge were in the golden mean ratio to form the vertices of an icosahedron **Figure 8c**.



a **b** **c**

Figure 8: Tetrahedron (**a**), Octahedron (**b**) and Icosahedron (**c**).

Faces of a tetrahedron could be rotated either clockwise or counter-clockwise to produce an identical octahedron. However, when the faces of this octahedron were rotated in a clockwise direction an icosahedron **Figure 9c** was derived. When these vertices were rotated in a counter-clockwise direction a icosahedron with a different orientation **Figure 9d** was derived. When the **Figure 9c** and **Figure 9d** are displayed simultaneously with **Figure 9b**, the direction of the rotation can be more easily seen. When **Figure 9a** is added to **Figure 9e**, the results can be seen in **Figure 9f** showing a full set of polyhedra.



a **b** **c** **d** **e** **f**

Figure 9: Tetrahedron (**a**), Octahedron (**b**), Chiral Icosahedron (**c&d**), & ***Snubs*** (**e&f**).

9. Two Snub Polyhedra Complete the Set of Archimedean Polyhedra

Six faces of a cube were scaled by ½, and rotated by 45 degrees for vertices of a cuboctahedron . These six faces were further rotated and scaled for the twenty-four vertices of a snub cube. Similarly, the twelve faces of a dodecahedron were rotated by 36 degrees and scaled by ½ to form the vertices of an icosidodecahedron. These twelve faces were further rotated and scaled to form the sixty vertices of the snub dodecahedron. This ployhedron is also known as a snub icosahedron or a snub icosidodecahedron.

10. Statistical Study of Data Model Vertices

When I computed the data for Platonic, Archimedean, Prism, Anti-Prism and their Dual polyhedra, I printed 18 decimal digits tables for the double precision, floating point numbers. The tables consisted of the three coordinates of each vertex, a radius for each vertex, three coordinates of the mid-edges point, a radius for each mid-edge, three coordinates of the mid-faces point, and a radius for each mid-face point. I also computed and printed all edge lengths between vertices and all dihedral angles between faces.

11. Conclusions

When Haresh positioned the last polyhedron in his quasi-crystal structure, it registered with the previously positioned polyhedra without significant error. This result reinforced the precision of the polyhedra derived with this method of such simple computation. From the printed tables I could visually observe that the data was showing accuracy to 16 digits and occasionally 17 digits. Since I could observe their accuracy, these polyhedra exhibited no directional error and had the potential to be applied to many problems containing polyhedra. Throughout my derivations of all these polyhedra, I used consistent geometric relationships between polyhedra for each of the tetrahedral, octahedral and icosahedral families. All vertices simultaneously moved along edges by a constant value, all half edges were simultaneously rotated and scaled by constant values, and all faces were simultaneously rotated and scaled by constant values.

In my work with transformations the concept of *moving vertices along edges and across faces* for ***nesting*** polyhedra was illustrated using transparency and color for different polyhedra. When animated, these ***nested*** polyhedra were seen to have their vertices coincident with the edges and faces of an initial polyhedron. Also, the concept of *edge rotating and scaling* for ***dual*** polyhedra was also illustrated using transparency and color for different polyhedra. Edges of the initial polyhedron intersected with edges of a dual polyhedron. In addition, they were orthogonal to each other and this relationship could be seen very clearly.

Finally, I used transparency and color for the different polyhedra to illustrate *face rotating and scaling* for ***snub*** polyhedra. The tetrahedron, with an octahedron inside, and an icosahedron inside the octahedron, clearly shows a relationship between vertex symmetry of three, four and five in a single composite **Figure 9f**. Since the golden mean ratio along the edge of the octahedron existed there was a serendipitous nature to such a precise relationship. This relationship is perhaps more subtle than can be easily seen by most viewers.

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References

[1] Coxeter, H.S.M., Regular Polytopes, Dover Publications, Inc., NY, 1973.
[2] Pearce, Peter & Susan Pearce, Polyhedra Primer, Van Nostrand Reinhold Co., NY, 1978.