“Geometry” in Early Geometrical Disciplines: Representations and Demonstrations

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Abstract

This paper discusses various manifestations of geometry in early geometrical disciplines with reference to specific cases from the Islamic “Middle Ages”, a period of intense scientific activity falling intermediately between the initial reception of Greek scientific material in the early Islamic period (8th-9th centuries AD), and their subsequent diffusion within both Islamic and to European lands (12-13th centuries AD). The paper begins with the classification of mathematical sciences in ancient Greek and early Arabic sources, and proceeds with the identification and distinction of aspects of geometry such as geometrical “representation” and “demonstration” through a case study of specific geometrical disciplines. The case study covers sample problems from four early geometrical disciplines: optics, mechanics, surveying and algebra: optics and mechanics are subdivisions of plane and solid geometry in Aristotelian classifications, surveying and algebra are the respective subdivisions of each in early Arabic Classifications. The samples include geometrical representations (definitions, figures, models) and geometrical demonstrations (illustrations, constructions, proof), as representatives of a range of Arabic and Persian scientific sources from the Islamic Middle Ages.

Introduction

This paper treats some historical aspects of geometry, a subject with an outstanding presence in both the sciences and the arts, and it is divided into three sections. The first section introduces early disciplinary classifications and divisions through the well-known “*Catalogue”* of Al-Fārābī (d. ca. 950 AD)[1]. The focus is on geometrical disciplines such as optics and mechanics whose long entries under “mathematical sciences”, along with sub-entries such as surveying and algebra [2,3], reflect an extension of the classical “quadrivium” beyond the foursome of “arithmetic, geometry, astronomy and music”. The second section highlights distinctions between geometrical “representation” and “demonstration” through the discipline of geometry itself: examples of “geometrical representations” are presented as definitions and figures involving line, circle, and square and rectangle with reference to the *Elements* of Euclid (ca. 300 BC), one of the earliest and most circulated works in both Arabic and Persian [4]; examples of “geometrical demonstrations” are presented with reference to the formal divisions of a Euclidean proposition, itemized in Euclid’s first book as: enunciation, illustration, definition or specification, geometrical construction or set up, proof or demonstration and conclusion [5]. The distinction between “geometrical representation” and “geometrical demonstration” is further extended to “geometric” optics through specific geometrical “models” based on the *Optics* of Euclid: as geometrical representations such as angles and cones represent various ancient Greek and medieval Arabic visual models (e.g. Euclid and Ptolemy: 4thc. BC - 2nd c. AD; al-Kindī and Ibn al-Haytham: 9th-11th centuries)[6], demonstrations themselves represent a shift from the geometry-based illustrations of Greek antiquity to the physical set ups of the Islamic Middle Ages. The paper’s third and final section concludes with a presentation of four sample problems from optics, surveying, mechanics and algebra, and a discussion of their respective representations and demonstrations (Samples and Figures 1-4)[7-10].

1. Disciplinary Classifications

Synopsis: The Mathematical Sciences in the important and influential *Catalogue* of Al-Fārābī (d. ca. 950 AD) include, in addition to the four disciplines forming the basis of the classical "quadrivium" (arithmetic, geometry, astronomy, and music), three others: the sciences of optics, mechanics and weights. While optics and mechanics are classified as the respective subdivisions of plane and solid geometry since Aristotle, surveying and algebra appearing under the respective entries on optics and mechanics in the same *Catalogue*, may be considered "geometrical" from yet other standpoints.

2. Representation vs. Demonstration

Synopsis: An explicit distinction between geometrical representation and demonstration is not found in early historical sources; but the distinction is clear in texts as early as Euclid's *Elements of Geometry* and *Optics* (ca. 300 BC), and its multiple variations: here, representations appear in the form of definitions and figures involving a point, line, circle, square and other geometrical figures, and demonstrations and their components appear under the formal divisions of a Euclidean proposition: enunciation, illustration, definition or specification, geometrical construction or set up, proof or demonstration and conclusion.

3. Case Studies

Synopsis: The following four samples provide case studies of the different manifestations and functions of geometry in disciples such as optics, mechanics, surveying and algebra. These include geometrical representations (definitions, figures, models), and demonstrations (illustrations, constructions, proofs), the latter with components such as geometrically based calculations (proportionality of triangles), and superimpositions (application of areas).

Sample 1: Optics (Euclid's *Optics*, Arabic Prop. 10, tr. early 9thc.: Kheirandish, 1999, v.1: 30-32)
Right-angled Figures Viewed from a Far Distance

The first sample is an optical problem about the circular appearance of far rectangular objects, which going back to many classical sources, appears as the first of a few the problems included in Al- Fārābī's "optics" entry in his widely circulated *Catalogue*. The geometrical representations are visual rays and angles, and the geometrical demonstration is a Euclidean style proof based on other proofs and assumptions (Sample 1, Figure 1).

Sample 2: Surveying (Euclid's *Optics*, Arabic Prop. 20, tr. early 9thc.: Kheirandish, 1999, v. 1: 58-60)
Determination of Heights Through Reflecting Visual Rays

The second sample is a surveying problem about the determination of the height of an object by means of a plane mirror. Problems of surveying such as determinations of height, width and depth are also included in Al-Fārābī's entry, but here the use of a plane mirror makes this a problem of indirect, rather than direct vision. The geometrical representations are, again, visual rays and angles, and the geometrical demonstration, a Euclidean style proof, this time based on the proportionality of two similar triangles for the calculation of unknown quantities based on known ones (Sample 2, Figure 2).

Sample 3: Mechanics (Heron's *Mechanics*, Arabic tr.9thc.: Kheirandish, work in progress)

Lesser Weights Lifting Greater Weights

The third sample is a mechanical problem as early as the Pseudo-Aristotelian *Mechanical Problems* (847a24-847b: ca. 4th c. BC, Anonymous Arabic translation: 9thc.?), which is set in terms of the "wondrous" effect of small weights lifting great weights through mechanical devices (mechané). This is also a central problem in the *Mechanics* of Heron of Alexandria (Book II: 2ndc. AD, Arabic translation: Qusṭā ibn Lūqā, d. ca. 912-13), there expressed in terms of "powers" that include, in addition to the "wheel and axle", "pulley", "wedge" and "screw", the "lever", one whose calculation is described in terms of the ratio of the load to be moved and the force meant to move it. The geometrical representation of a mechanical device such as the lever is through geometrical shapes and relations of lines, angles, and triangles, and the geometrical demonstration of the function of the mechanical principle itself is based on the proportionality of similar triangles (Sample 3, Figure 3).

Sample 4: Algebra (Euclid's *Elements*, Book II. Prop. 4, tr. early 9thc.: Kheirandish, 1987: 18-19)
Algebraic Identity Corresponding to a Quadratic Equation

The fourth sample is a problem that appears as the fourth proposition of the second book of Euclid's *Elements*, and is treated as a more general problem as early as in a ninth-century Arabic text by Thābit ibn Qurra (d. ca. 901). While the identity in Euclid's Book I (I. 47) corresponds to the problem known as the "Pythagorean Theorem" where the square of the hypotenuse is equal to the sum of the square of the two sides, the problem in Book II (II.4) deals with an identity corresponding to a second degree algebraic equation where the square of the sum of two segments on the sides of a square is equal to the square of each, plus twice their products. The geometrical representations here are geometrical shapes such as lines, squares, and rectangles, and the geometrical demonstration is based on the application of the areas where the area of the larger square is equal to the area of the two smaller squares, plus the two side rectangles (Sample 4, Figure 4).

References

[1] Abū Naṣr Al-Fārābī, *Enumeration of the Sciences (Iḥṣā' al-'ulūm*), ed. 'Uthmān Amin [= Osman Amine], Cairo: Librairie Anglo-Égyptienne, pp. 93-110. 1968 (also, 1931, 1949).
[2] Al-Fārābī, "Science of Optics": tr. A. I. Sabra's *The Optics of Ibn al-Haytham*: Books I-III On Direct Vision, Translated with Introduction and Commentary, 2 vols. (London: The Warburg Institute, University of London), edited by J. B. Trapp, 2 vols., Vol. 2: lvi-lvii. 1989.
[3] Al-Fārābī, "Sciences of Mechanics" tr. George Saliba, "The Function of Mechanical Devices in Medieval Islamic Society", *Science and Technology in Medieval Society*, ed. Pamela O. Lang, *Annals of the New York Academy*, pp. 144-145. 1985; "Sciences of Weights" tr. Marshall Clagett, "Some General Aspects of Physics in the Middle Ages," *Isis*, 39, pp. 29-44: 32. 1948.
[4] John E. Murdoch, "Euclid," *Dictionary of Scientific Biography* (*DSB)*, 4, esp. pp. 438–443. 1971.
[5] Thomas Heath, *The Thirteen Books of Euclid's Elements,* 3 volumes, Vol. I, pp. 129-131. 1926.
[6] Elaheh Kheirandish, "The Arabic 'Version' of Euclidean Optics: Transformations as Linquistic Problems in Transmission," *Tradition, Transmission, Transformation*, eds. F. Jamil Ragep and Sally P. Ragep with Steven Livesey, (Leiden: Brill), pp. 227–247: 247. 1996.
[7] Elaheh Kheirandish, *The Arabic Version of Euclid's Optics*, 2 volumes, Springer-Verlag: Sources in the History of Mathematics and Physical Sciences, no. 16, Vol. 1, pp. 30-32. 1999.
[8] Elaheh Kheirandish, *The Arabic Version of Euclid's Optics*, 2 volumes, Vol. 1, pp. 58-60. 1999.
[9] Elaheh Kheiranish, "The Arabic and Persian Traditions of the *Mechanics* of Heron and Pappus of Alexandria " (Work in Progress).
[10] Elaheh Kheirandish: *Problems Corresponding to Quadratic Equations in Early Mathematical Texts: A Chapter in the History of Algebra* (unpublished Master's thesis), 1987, pp. 18-19.

Sample 1: Optics (Euclid's *Optics*, Arabic Proposition10) Right-angled Figures Viewed from a Far Distance
From: Elaheh Kheirandish, *The Arabic Version of Euclid's Optics,* 2 vols., Springer-Verlag, 1999, vol. 1, pp. 30-32:

The enunciation is that right-angled figures when seen from a distance are seen as circular. The illustration is that of a right-angled figure (ABGD)(Figure 1), and the eye, that when at a distance, its visual rays do not pause at a single point but shift, such that what is seen is figure EZHTKLMN but not what is in between, namely corners. The conclusion is that because what is hidden from sight, namely the object's angles, is the part of the figure with least magnitude, and that any figure with no angles is circular, then all figures are seen from a far distance as circular, and so is the right-angled figure, ABCD.

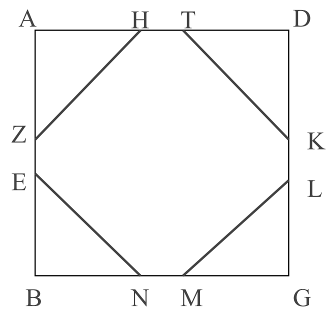

Figure 1

Sample 2: Surveying (Euclid's *Optics*, Proposition 20) Determination of Heights Through Reflecting Visual-rays
From: Elaheh Kheirandish, *The Arabic Version of Euclid's Optics*, Springer-Verlag, 1999, vol. 1, pp. 58-60.

The problem is to determine the height of an object by means other than the sun. The illustration is that of a point and two lines, representing the eye (G), an upright object (AB), and a horizontal plane mirror (ED)(Figure 2). The geometrical construction is that of line EB, of a visual ray GH, deflecting at point H to fall on the farthest point of AB (A) such that a line from H to T meets a perpendicular from the eye (G), and the deflection of ray GH at H at equal angles results in two similar triangles, AHB and GHT. The conclusion is that because the ratio of HB to AB is known, from the measurable values of GT and TH, the magnitude of AB can be determined.

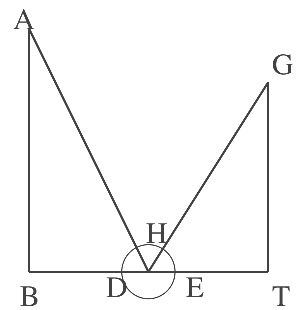

Figure 2

Sample 3: Mechanics (Aristotelian Mechanical Problems) Lesser Weights Lifting Greater Weights (Heron Book II)
From: Elaheh Kheirandish, "The Arabic and Persian Traditions of the Mechanics of Heron" (Work in Progress).

The five powers or simple machines, described by Heron, are the wheel and axle, pulley, lever, wedge and screw.
In the case of the lever a heavy load (A) is moved by a little force (C), acting on it through the fulcrum (B) (Figure 3).
According to Heron, the size of such a machine is set up according to the size of the load to be moved with it, and its calculation take place according to the ratio of the load one wants to move to the force that is meant to move it.

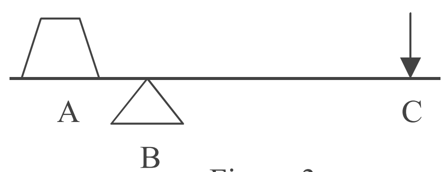

Figure 3

Sample 4: Geometric Algebra (Euclid's *Elements*, II. 4) Algebraic Identity Corresponding to Quadratic Equation
From: Elaheh Kheirandish: *Problems Corresponding to Quadratic Equations in Early Mathematical Texts* (unpublished), 1987, pp. 18-19.

The problem in Book II (II.4) deals with an identity that corresponds to a second degree algebraic equation, where the square of the sum of two segments on the sides of a square is equal to the square of each, plus twice their products (Figure 4). Similar to the problem, where the square drawn on the hypotenuse of a right-angled triangle is equal to the sum of those drawn on the two sides, the demonstration in this proposition is through the equality of areas involving an outer square and its inner areas such that the outer square area $(a+b)^2$ is equal to the sum of the inner areas a^2+b^2+2ab (C cuts AB at random: AC=a; CB=b).

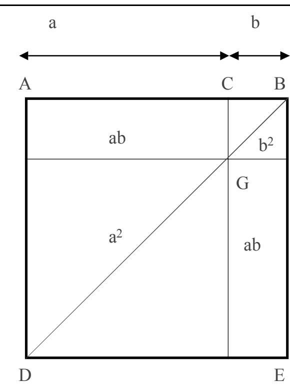

Figure 4