Portraits of Groups

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Abstract

This paper represents some small finite groups as groups of transformations of a compact surface of small genus. In particular, we start with a designated pair of regions of this surface and each region is labeled with the group element, which transforms the designated region into it. This gives a portrait of that finite group. These surfaces and the regions corresponding to the group elements are shown in this paper. William Burnside first gave a simple example of such a portrait in his 1911 book, "Theory of Groups of Finite Order".

Introduction and Historical Perspective

There are many ways to draw a picture of a finite group. One possibility is to let the group elements be represented by one to one transformations of the points of a surface. This idea was developed by Dyck [3] and elaborated further in Burnside [1]. Burnside started with circles in the plane and the transformation was inversion in the circle. Inversion in a circle can be defined in a Euclidean plane with a "point at infinity" appended. The plane with a "point at infinity" can be identified with the Riemann sphere, Σ. It can be shown that inversion in circle C is given by the equation $I_C(z) = -\dfrac{\bar{b} \cdot \bar{z} + c}{a \cdot \bar{z} + b}$, where C has equation $az\bar{z} + bz + \bar{b}\bar{z} + c = 0$ with a and c real and b complex. This map is an anti-automorphism of the Riemann sphere (Jones and Singerman [4], p. 29).

The group generated by these transformations is determined by the relationship between the circles. For example, starting with a circle and a straight line tangent to it (a circle of infinite radius), the set of inversions of the circles in each other gives the diagram given in Figure 1 (Burnside [1], p. 377). The transformation, S is given by composing first a reflection in the line and then an inversion in the circle. The plane is divided into black and white regions as in figure 1 and each transformation takes the white regions into themselves and the black regions into themselves. If we start with a white region labeled E for the identity, then the region into which E is transformed by S^n can be labeled by that group element. This gives a nice graphical picture of the integers as a group of transformations.

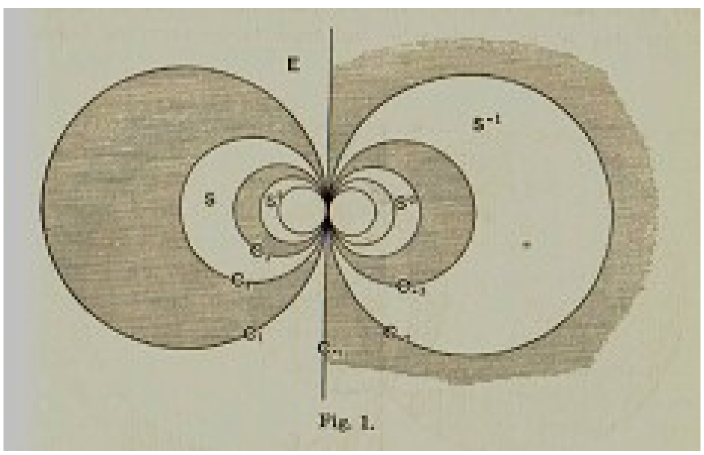

Figure 1, Burnside [1], Page 377.

The same ideas are used in Burnside [1, p. 379] to construct a free group on n generators, F_n. This construction fills up a unit disk with black and white regions and the transformations are given in the same way. We have used Geometer's SketchPad to reconstruct part of this portrait of a free group on two generators (Figure 2). This figure is very similar to the figure in Burnside [1], Page 380. Each "triangle" is bounded by arcs colored red, blue or black in our sketch. Inversion in any single arc will take a shaded region into a non-shaded region and vice versa. Therefore, each group action is represented by the composite of two such inversions. Inversion through first a red arc and then a blue arc corresponds to multiplying on the left by the generator S. Multiplying on the left by the generator T corresponds to inversion through black and then red. Multiplying on the left by ST corresponds to inversion through black and then blue. If we considered inversion through a black arc first and then a blue arc as the inverse of a single generator, R, then we could interpret this picture as a portrait of a group with presentation $\langle r,s,t \mid rst=1\rangle$.

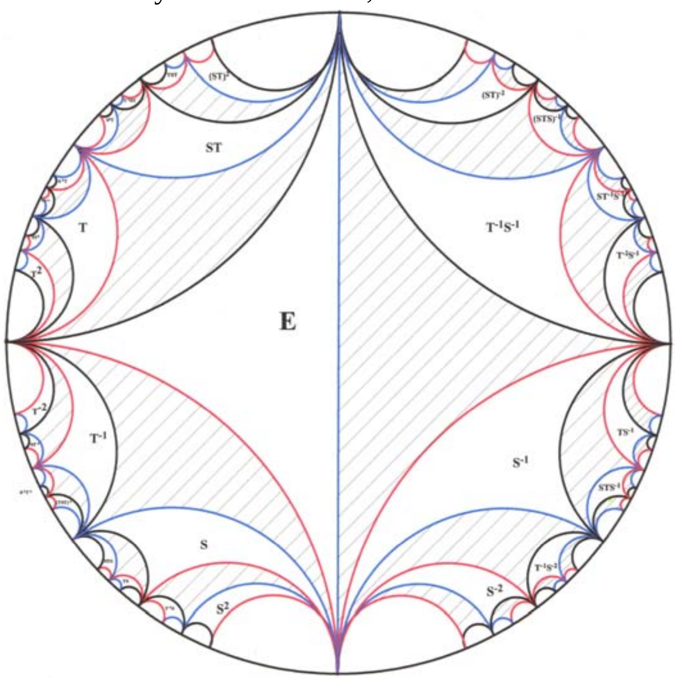

Figure 2, Portrait of a Free Group

Now suppose that we have a finite group, G, generated by n generators. This group is the image of F_n by a normal subgroup, N. After associating an element of F_n to each region, the final step is to identify all regions with labels from the subgroup, N. After this identification, we have the finite group, G, represented as a group of transformations on a surface of some genus. For n = 3, this is really the image of a quotient of a triangle group, $\Gamma(u,v,w)=\langle r,s,t \mid r^u=s^v=t^w=rst=1\rangle$. The transformation of inversion in a circle is an anti-analytic transformation of the Riemann sphere into itself. Therefore, any group represented in this way acts on a Riemann surface in an orientation-preserving way. The next section will attempt to give some portraits of small groups.

Group Portraits

Every compact Riemann surface with genus g is topologically equivalent to a sphere with g handles or equivalently, a sphere with g holes in it. This surface may be drawn and colored with white and black regions that represent a finite group of transformations, which act on the surface. Figure 3 gives Burnside's example of the picture of the group of quaternions, Q. The quaternions are the smallest group of strong symmetric genus 2. They also have a presentation as an image of the triangle group $\Gamma(4,4,4)$. This is a very symmetric presentation and that makes it easy to construct the portrait. The other groups of strong symmetric genus 2 are the dicyclic group, DC_3, the quasidihedral group, QD_4, the group $(4,6 \mid 2,2)$, using notation from Coxeter and Moser [2], $SL(2,3)$ and $GL(2,3)$ (See May and Zimmerman [5]). We consider the dicyclic group, DC_3 next.

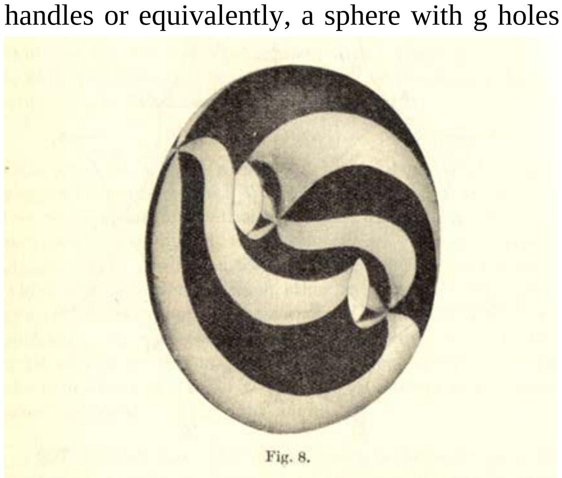

Figure 3, Burnside [1], Page 396.

The dicyclic group of order 12 has presentation $\langle x, y \mid x^6 = 1, x^3 = y^2, y^{-1}xy = x^{-1} \rangle$. Its genus action is given by its presentation $\langle s, t \mid s^4 = t^4 = (st)^3 = 1, st = (ts)^2 \rangle$ as the image of the triangle

Figure 4 – Portrait of the Dicyclic Group of Order 12

group $\Gamma(3,4,4)$. The portrait of this group on a surface of genus 2 is shown in Figure 4. A red, a blue and a black arc of a circle bound each "triangle". The portrait consists of 12 white and 12 black triangles and has 10 vertices. The triangular regions that meet at a vertex are labeled in such a way that each white region is related to the adjoining white regions by multiplication on the left by either S, T or ST or its inverse. Therefore, each vertex could be classified as an S-vertex, a T-vertex or an ST-vertex depending on the labeling of its bounding regions. Since S and T have order 4, each S or T vertex has degree 8. Since the product ST has order 3, each ST vertex has degree 6. There are 3 S-vertices and the edges incident to them are blue and red. Similarly, there are 3 T-vertices and the edges incident to them are black and red. Finally, there are 4 ST-vertices and the edges incident to them are blue and black. This gives 10 vertices, 36 edges and 24 faces and so the Euler characteristic is –2.

Finally, we construct the portrait of the "quasiabelian" group, $QA_4 =< 2,2 \mid 2 >$ of order 16. This is a group of strong symmetric genus 3; it is also the rotation group of a regular map of genus 3. This group has presentation $< x, y \mid x^8 = y^2 = 1, yxy = x^5 >$ and it is the image of the triangle group $\Gamma(2,8,8)$. Its presentation as an image of $\Gamma(2,8,8)$ is $< s, t \mid s^8 = t^8 = (st)^2 = 1, s^2 = t^2 >$. The portrait consists of 16 white and 16 black triangles and has 12 vertices. Since S and T have order 8, each S or T vertex has degree 16. Since the product ST has order 2, each ST vertex has degree 4. There are 2 S-vertices, 2 T-vertices and 8 ST-vertices. Each ST-vertex connects only to the S and the T vertices. Each S or T vertex connects 4 times to the same S vertex, 4 times to the same T vertex and to each one of the ST vertices. This results in the portrait in Figure 5.

Figure 5 – Portrait of the Quasiabelian Group of Order 16

This gives 12 vertices, 48 edges and 32 faces and so the Euler characteristic is –4. Therefore, it may be drawn on a surface of genus 3. A surface of genus 3 may be constructed in many topologically equivalent ways. I have chosen a way where the symmetry of the construction complements the structure of the group.

References

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