Polyhedral Transformation: *Explosion-Implosion*

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Abstract

Polyhedral transformation software continuously produced polyhedron. In other words, for different inputs, a different polyhedron was output. For some, the output was a Platonic or Archimedean polyhedron. The transformation was modeled as a fundamental region of a polyhedron, and when polygons from a fundamental region were combined with symmetry transformations of reflections and rotations, a complete polyhedron was formed. These complete polyhedron originated from the tetrahedral, octahedral, and icosahedral families. With the use of animation, the polyhedron were transformed smoothly and continuously from one polyhedron to another. For example, an icosahedron was transformed into a dodecahedron.

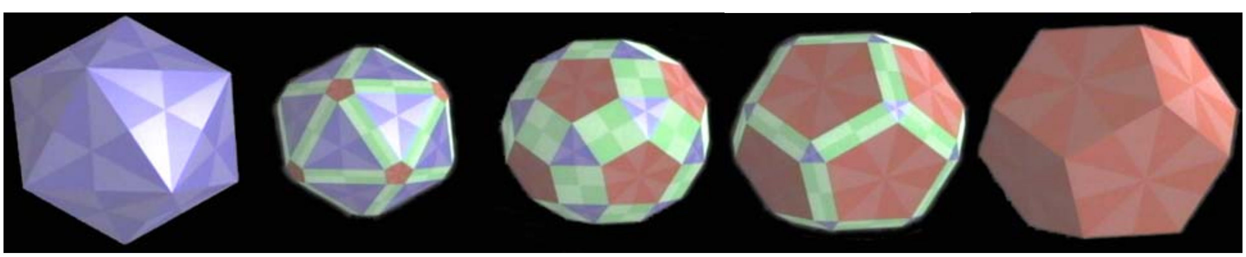
1. Introduction

In 1982 I attended an exhibit by Haresh Lalvani at Pratt Institute in Brooklyn, New York. Haresh showed his work on polyhedral transformations. Many polyhedron models were displayed along with words and diagrams describing the transformations. This work appeared in his Ph.D. dissertation completed with Buckminster Fuller at the University of Pennsylvania [1].

He exhibited a polyhedral transformation he referred to an *explosion-implosion*. In this paper I will start with an example of an *explosion-implosion*. I will follow this example by describing a polyhedron's *fundamental region*, which is a minimal *region* for a polyhedron bounded by symmetry planes [2, p. 63]. I will describe input, output and an implementation for software to model *explosion-implosion.* This implementation includes examples of polyhedron referenced by the eight vertices of a unit edge reference cube. A ten minute animation accompanies this paper to show the continuous three-dimensional nature of *explosion-implosion*.

2. *Explosion-Implosion* Example

Expolsion-implosion applies simultaneously to every vertex, edge, and face of a polyhedron. An *explosion-implosion* example transforms an icosahedron, in **Figure 1a**, to a dodecahedron in



a **b** **c** **d** **e**

Figure 1: Icosahedron (**a**), rhombicosidodecahedron (**c**), and dodecahedron (**e**).

Figure 1e, through a rhombicosidodecahedron in **Figure 1c**. The intermediate polyhedron, in **Figure 1b,** has pentagonal faces that are *explosions* of the icosahedron vertices. Simultaneously occuring with these *explosions* are *explosions* of the edges of the icosahedron into rectangular faces, in **Figure 1b**. These rectangular faces continue to *explode* until they become square faces of a rhombicosidodecahedron, in **Figure 1c**. Now the *implosions* start where these square faces become rectangles and the triangles become smaller. At the same time, pentagons are *exploding* to become larger, in **Figure 1d**. The *implosion*s continue until the rectangles have become edges and the triangles become vertices, resulting in the dodecahedron, in **Figure 1e**. The edges of the icosahedron are orthogonal to the edges of the dodecahedron and the rectangles, in **Figure 1b,** are orthogonal to the rectangles in **Figure 1d**. *Explosion-implosion* produces unit edge polyhedron. Thus, both the icosahedron and the dodecahedron have unit edge lengths. The rhombicosidodecahedron, an Archimedean polyhedron, is an intermediary of this *explosion-implosion.* As a result, it has edge lengths of ½. The polyhedron, in **Figure 1b**, **1c**, and **1d,** are all topological equivalents of each other. *Explosion-implosion* can be seen in an accompanying 10 minute animation

Explosion-implosion uses a unit edge reference cube, in **Figure 2**. Each point in this reference cube refers to a different polyhedron. The three individual coordinates of a reference

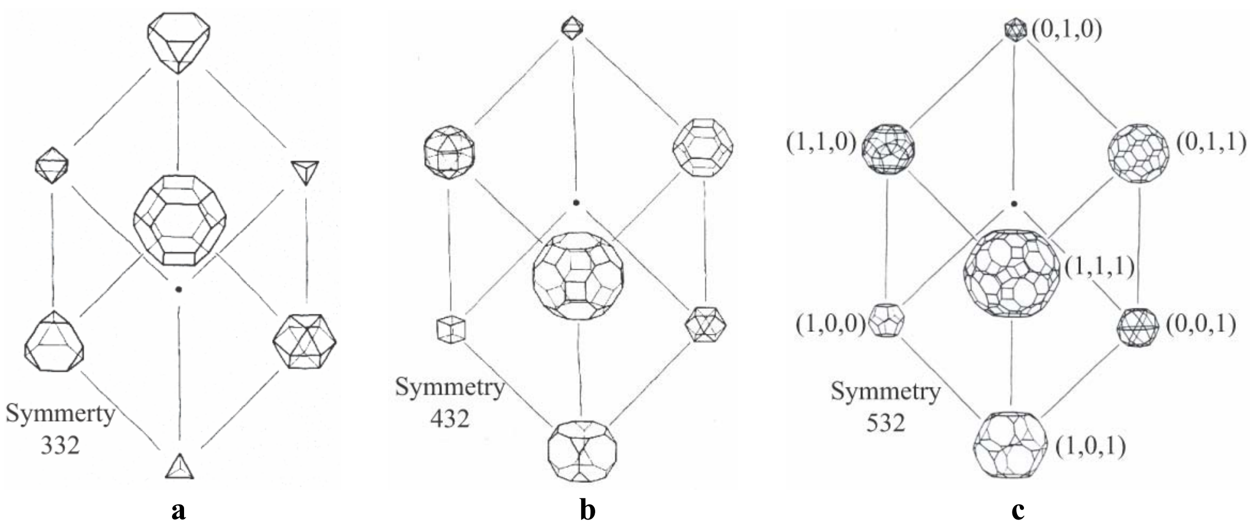
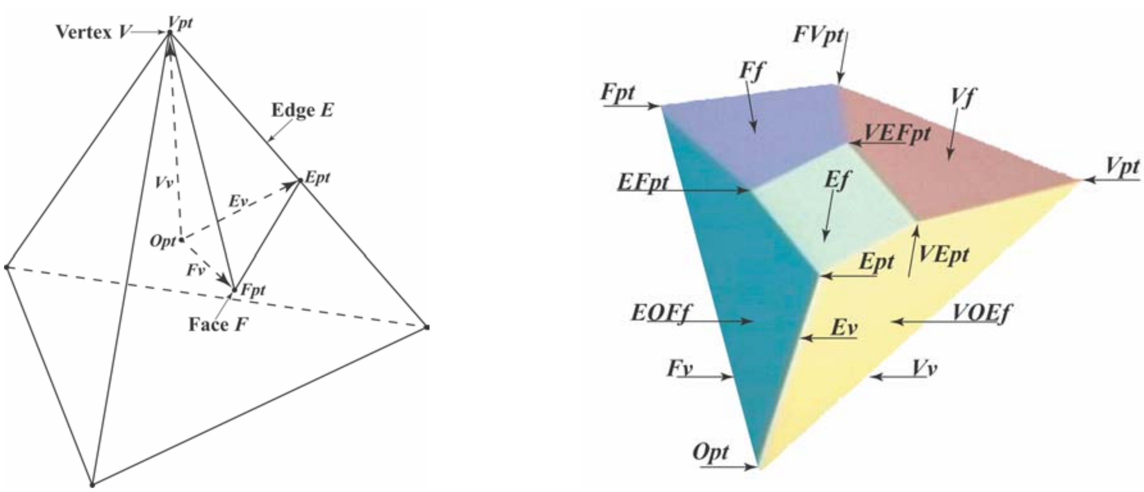


Figure 2: A unit edge reference cube of tetrahedral (**a**), octahedral (**b**), icosahedral (**c)** families.

point are equal to the three different edge lengths of the derived polyhedron for that point. For example, the point (0,1,0) refers to a unit edge icosahedron, in **Figure 2c,** and (1,0,0) refers to a unit edge dodecahedron. Vertex (0,0,0) is a reference point for a null polyhedron. Vertices (0,0,1), an icosidodecahedron, (1,1,0), a rhombicosidodecahedron, (0,1,1), a truncated icosahedron, (1,0,1), a truncated dodecahedron and (1,1,1), a truncated icosidodecahedron, are all polyhedron with unit edge length. The polyhedron, in **Figure 1c,** has a reference point of (½,½,0) and an edge lengths of ½. *Explosion-implosion* in this paper produced polyhedron from tetrahedral, **Figure 2a**, and octahedral families, **Figure 2b,** [1, pp. 9,10], and additionally from the icosahedral family, in **Figure 2c**.

3. *Fundamental Region* of a Polyhedron.

A *fundamental region* of a tetrahedron, in **Figure 3a,** was defined by three vectors ***Vv***, ***Ev***, and ***Fv***, which originate at the center of the tetrahedron, ***Opt***. Vector ***Vv*** terminated at a vertex ***Vpt***. Vector ***Ev*** terminated at an edge ***E*** in a point ***Ept*** and is orthogonal to that edge. A third vector ***Fv*** terminated orthogonally to a face ***F*** at point ***Fpt***. The three vectors ***Vv***, ***Ev***, and ***Fv*** were equal to the radii of an *outer* sphere of its vertices, the *mid* sphere orthogonal to its edges, and the *inner* sphere orthogonal to its faces. **Figure 3b** was a general *fundamental region* with three faces ***Vf***, ***Ef***, and ***Ff***, and two visible symmetry faces ***VOEf*** and ***EOFf***. A third symmetry face ***FOVf*** is not visible in **Figure 3b.** Other points of interest in this *fundamental region* were ***VEpt***, ***EFpt***, ***FVpt***, and ***VEFpt***, where face planes and side symmetry planes intersect.



a **b**

Figure 3: *Fundamental region* line drawing (**a**) and shaded polygons (**b**).

3. Inputs for *Explosion-Implosion*

3.1. Symmetry Numbers for a Regular Polyhedron. Symmetry numbers for a regular polyhedron, in **Figure 4**, refer to the symmetry of its vertices ***sv***, the symmetry of its edges ***se***, and the symmetry of its faces ***sf***. Each of these symmetry numbers were integer inputs for *explosion-implosion* of regular polyhedron.

Family	**sv**	**se**	**sf**
Tetrahedral	3	2	3
Octahedral	4	2	3
Icosahedral	5	2	3

Figure 4: Symmetry Numbers for Families of Polyhedron.

3.2. Symmetry Transformations for a Polyhedron. A set of transformations composed of reflections and rotations were used to transform the faces of a fundamental region into a complete polyhedron. The tetrahedral family has 24 transformations, the octahedral family has 48 transformations, and the icosahedral family has 120 transformations.

The icosahedron of **Figure 1a** can be used to help understand a set of symmetry transformations. One triangle of the icosahedron face is composed of six sub-triangles, three light and three dark. Each of these sub-triangles is a face that has been transformed from a *fundamental region*. The first transformation is a reflection of a *fundamental region* through a side symmetry plane to form a second part of the face (i.e. one light and one dark triangle). This part of a face can be rotated twice to form a complete face (3 light & 3 dark triangles for a complete triangle). This composite face can be rotated four times to form a *lune* of triangles from the top to the bottom of the icosahedron. This *lune* of four triangles was rotated five times to form a complete polyhedron. The accompanying video illustrates this constructive approach to building a complete polyhedron from a *fundamental region*.

Explosion-implosion software contained four tables of 4x4 arrays of the values for the four sets of symmetry transformations. Depending on the input of the symmetry numbers, a different table or set of symmetry transformations was used to display that complete polyhedron

3.3. A Point in a Unit Edge Cube. A unit edge cube provides a three-dimensional reference space for a point whose three coordinate values ***pv***, ***pe***, and ***pf***, are real inputs (0 <= ***pv, pe, pf*** **<=** 1) for *explosion-implosion*, in **Figure 2**. Inputs ***pv***, ***pe***, and ***pf*** will produce a polyhedron with edge lengths equal to these three input values.

4. Output from *Explosion-Implosion*

The three faces ***Vf***, ***Ef***, and ***Ff*** of the *fundamental region*, in **Figure 3b**, are the output from *explosion-implosion* software. These three faces were derived by solving for seven points, ***Vpt***, ***Ept***, ***Fpt***, ***VEpt***, ***EFpt***, ***FVpt***, and ***VEFpt***, in **Figure 3b**. When these three faces are coupled with an appropriate set of transformations, a complete polyhedron was displayed as output. **Figure 1** shows five examples of complete polyhedron output derived from a *fundamental region*.

5. Implementation of *Explosion-Implosion*

5.1. Concept A clear explanation of *explosion-implosion* polyhedral transformation can be illustrated as three face planes orthogonal to and sliding up and down the three symmetry vectors of the *fundamental region*.

5.2. Symmerty Numbers Once symmetry numbers ***sv***, ***se***, ***sf*** were input, angles between symmetry vectors ***Vv***, ***Ev***, and ***Fv***, and direction cosines for the three face planes ***Vf, Ef, and Ff*** were derived. These derivations appear in Coxeter [2]. A constant value for the length of a half edge, ***l,*** of a polyhedron was consequently provided to this software for unit edge polyhedron. The value of ***l*** was 0.5.

5.3. Symmetry Vectors The three symmetry vectors ***Vv***, ***Ev*** and ***Fv***, were combined in pairs to form three angles. Vectors ***Vv*** and ***Ev*** form angle ϕ, where ***Ev*** and ***Fv*** form φ, and ***Fv*** and ***Vv*** form χ. For this implementation some interim values were defined; ***pisf*** and ***pisv*** are examples of such interim values using inputs ***sv*** and ***sf***.

$$pisv = \pi/sv, \qquad pisf = \pi/sf$$

$$\phi = acos(\ csc(pisv) * cos(pisf)\ )$$

$$\varphi = \mathrm{acos}(\ \cos(pisv) * \csc(pisf)\ )$$

$$\chi = \mathrm{acos}(\ \cot(pisv) * \cot(pisf)\ )$$

5.4. Three Polyhedron from a Symmetry Family. The length of radii for an *outer* sphere , a *mid* sphere, and an *inner* sphere were derived for each of three different polyhedron for each symmetry family, in **Figure 5**. The three polyhedron were the regular-faced polyhedron, its dual, and their intersection polyhedron. These radii were derived so that each of the subsequently derived polyhedron had a unit edge length.

Family	Regular	Intersection	Dual
tetrahedral	tetrahedron	octahedron	tetrahedron
octahedral	octahedron	cuboctahedron	cube
icosahedral	icosahedron	icosidodecahedron	dodecahedron

Figure 5: Regular, Dual, and Intersection Polyhedron for Symmetry Families.

The value of h is the sides of an equatorial polygon for the intersection polyhedron [2, p19].

$$pih = \mathrm{acos}(\ \mathrm{sqrt}(\ \cos(pisv) * \cos(pisv) + \cos(pisf) * \cos(pisf)\ )\ )$$

The regular-faced polyhedron radii for its *outer* sphere rv, its *mid* sphere re, and its *inner* sphere rf, were derived as follows.

$$rv = l * \sin(pisv) * \csc(pih)$$
$$re = l * \cos(pisf) * \csc(pih)$$
$$rf = l * \cot(pisf) * \cos(pisv) * \csc(pih)$$

The dual of the regular-faced polyhedron radii for its *outer* sphere dv, its *mid* sphere de, and its *inner* sphere df, were deriveded as follows.

$$dv = l * \cot(pisv) * \cos(pisf) * \csc(pih)$$
$$de = l * \cos(pisv) * \csc(pih)$$
$$df = l * \sin(pisf) * \csc(pih)$$

The intersection polyhedron of the regular-faced polyhedron and its dual polyhedron radii for its *outer* sphere iv, its *mid* sphere ie, and its *inner* sphere if, were derived as follows. Vertex radii csv and csi for two polygons were first derived from vertex and face symmetry numbers [2, p3]. Radii for the intersection polyhedron used angles ϕ and φ from the symmetry vectors.

$$csv = l * \csc(pisv)$$
$$csf = l * \csc(pisf)$$
$$iv = \cot(\phi) * csv$$
$$ie = \csc(\phi) * csv$$
$$if = \cot(\varphi) * csf$$

The equations for the angles between the symmetry vectors and the length of the radii for a regular-faced polyhedron and its dual appear in Coxeter [2]. The radii for the intersection polyhedron, were derived from equations appearing in Coxeter [2].

5.5. Nine Radii Multiplied By a Point Yield Three Radii. The nine radii just derived formed a 3x3 matrix ***Rm*** in **Figure 6**. Three coordinate values ***pv***, ***pe***, and ***pf***, from **3.3**, formed a vector ***Pv***. Multiplying ***Rm***, by ***Pv***, yields a vector ***Fv*.** Values ***v***, ***e***, and ***f*** of ***Fv*** are the radii of the *outer* sphere, the *mid* sphere, and the *inner* sphere for an output polyhedron.

$$\begin{vmatrix} v \\ e \\ f \end{vmatrix} = \begin{vmatrix} rv & iv & dv \\ re & ie & de \\ rf & if & df \end{vmatrix} * \begin{vmatrix} pv \\ pe \\ pf \end{vmatrix}$$

Figure 6: ***Fv = Rm * Pv***

5.6. Points and Planes in a Fundamental Region. Spherical coordinates were conveniently formed from the three radii, and also the symmetry angles for the spherical points ***Vsp***, ***Esp***, and ***Fsp***.

Spherical Angles

$$vth = 0 \;,\; vph = 0$$
$$eth = \phi \;,\; eph = 0$$
$$fth = \chi \;,\; fph = pisv$$

Spherical Points

$$Vsp = [\, vth , vph , v \,]$$
$$Esp = [\, eth , eph , e \,]$$
$$Fsp = [\, fth , fph , f \,]$$

Spherical coordinates were chosen so that the center of the polyhedron was at the origin. The spherical points were converted into Cartesian points ***Vpt***, ***Ept***, and ***Fpt***, in **Figure 3b**.

Cartesian Points

$$Vpt = [\, \sin(vth) * \sin(vph) , \cos(vth) , \sin(vth) * \cos(vph) \,]$$
$$Ept = [\, \sin(eth) * \sin(eph) , \cos(eth) , \sin(eth) * \cos(eph) \,]$$
$$Fpt = [\, \sin(fth) * \sin(fph) , \cos(fth) , \sin(fth) * \cos(fph) \,]$$

These Cartesian point coordinate values were the direction cosines for the three planes ***Vpl***, ***Epl***, and ***Fpl***, and the three radii ***v***, ***e***, and ***f*** were distances from the origin.

Face Planes

Vpl **vector from [*Vpt.x* , *Vpt.y* , *Vpt.z* , *-v*]**
Epl **vector from [*Ept.x* , *Ept.y* , *Ept.z* , *-e*]**
Fpl **vector from [*Fpt.x* , *Fpt.y* , *Fpt.z* , *-f*]**

The three points ***Vpt***, ***Ept***, and ***Fpt*** were combined with the origin point ***Opt*, to** determine vectors for the three symmetry planes ***VOEpl***, ***EOFpl***, and ***FOVpl***.

Symmetry Planes

VOEpl **vector from [*Vpt* , *Opt* , *Ept*]**
EOFpl **vector from [*Ept* , *Opt* , *Fpt*]**
FOVpl **vector from [*Fpt* , *Opt* , *Vpt*]**

5.7. Points for Faces of the *Fundamental Region*. The six planes, ***Vpl***, ***Epl***, ***Fpl***, ***VOEpl***, ***EOFpl***, and ***FOVpl***, intersect in sets of three to yield the seven points for faces of the *fundamental region*. Three of these seven points, ***Vpt***, ***Ept,*** and ***Fpt***, in **Figure 3b,** are at the corners of the *fundamental region* in line with the symmetry vectors ***Vv***, ***Ev***, and ***Fv***. These three points are at the intersection of two side symmetry planes and one face plane.

Fundamental Region Corner Points
***Vpt* from intersection of planes [*VOEpl* , *Vpl* , *FOVpl*]**
***Ept* from intersection of planes [*EOFpl* , *Epl* , *VOEpl*]**
***Fpt* from intersection of planes [*FOVpl* , *Fpl* , *EOFpl*]**

Three more points, ***VEpt***, ***EFpt***, and ***FVpt***, are on the sides of the fundamental region. These three points are at the intersection of two face planes and one side symmetry plane. The seventh point ***VEFpt***, is formed by the intersection of the three face planes. This ***VEFpt*** point moved over the interior, as well as the boundary of the *fundamental region* .

Fundamental Region Side Points
***VEpt* from intersection of planes [*Vpl* , *VOEpl* , *Epl*]**
***EFpt* from intersection of planes [*EpI*, *EOFpl* , *Fpl*]**
***FVpt* from intersection of planes [*Fpl* , *FOVpl* , *Vpl*]**
Fundamental Region Face Point
***VEFpt* from intersection of [*Vpl* , *Epl* , *Fpl*]**

5.7. Edges in Fundamental Region. *Fundamental region* edges were formed by pairs of points. The lengths of these three edges ***Veg***, ***Eeg***, and ***Feg***, were precisely equal to the three coordinate values ***pv***, ***pe***, and ***pf***, of reference point ***Ppt***.

Fundamental Region Edges
***Veg* from points (*VEFpt* , *EFpt*)**
***Eeg* from points (*VEFpt* , *FVpt*)**
***Feg* from points (*VEFpt* , *VEpt*)**
Length of Edges
***pv* = length of (*Veg*)**
***pe* = length of (*Eeg*)**
***pf* = length of (*Feg*)**

5.8. Polyhedron of Points of the Reference Cube. Now I will use points from the reference cube as examples for polyhedron derived from *explosion-implosion.* The origin (0, 0, 0), of the reference cube, when multiplied by the radius matrix ***Rm***, yields radii (0, 0, 0), for a null polyhedron, in **Figure 2**. The reference cube corners, when multiplied by the radius matrix ***Rm***, yield radii vectors (***rv, re, rf***), (***iv, ie, if***), and (***dv, di, df***). These were the radii for a regular-faced polyhedron, its intersection polyhedron, and its dual, in **Figure 7**, and are the same as **Figure 5**.

corner ->	**(1, 0, 0)**	**(0, 1, 0)**	**(0, 0, 1)**
radii ->	**(*rv*, *re*, *rf*)**	**(*iv*, *ie*, *if*)**	**(*dv*, *de*, *df*)**
family	**regular**	**intersection**	**dual**
tetrahedral	tetrahedron	octahedron	tetrahedron
octahedral	octahedron	cuboctahedron	cube
icosahedral	icosahedron	icosidodecahedron	dodecahedron

Figure 7: Reference Cube Unit Vector Polyhedra.

The other four corners of the reference cube, when multiplied by the radius matrix ***Rm***, yield radii lengths for other unit edged polyhedron from the Archimedean family of polyhedron, in **Figure 8**. Prefix tr is used in **Figure 8** for truncated and the hedron suffix is dropped.

corner ->	**(1, 1, 0)**	**(0, 1, 1)**	**(1, 0, 1)**	**(1, 1, 1)**
tetraherdal	tr tetra	tr octa	tr tetra	cubocta
octaherdal	tr octa	tr cubocta	tr cube	rhombic cubocta
icosaherdal	tr icosa	tr icosidodeca	tr dodeca	rhombic icosidodeca

Figure 8: Reference Cube Other Corner Polyhedra.

Now that each of the polyhedron at the corners of the reference cube are defined, the polyhedron of the edges, faces, and interior of the reference cube can be considered. Along an edge of this cube is a polyhedron that is a combination of the polyhedron at the two corners for that edge. Similarly, on a face of this cube there is a polyhedron that is a combination of the polyhedron that are at four corners of that face. When considering the interior of the reference cube, a polyhedron that is a combination of the polyhedron that exits at the eight corners of the reference cube, is topologically equivalent to the (1, 1, 1) polyhedron.

6. Conclusion

Explosion-implosion software produced polyhedra that were structures in three dimensional space, and each polyhedron had three integer symmetry numbers for its vertices, edges, and faces. Each polyhedron was referenced by a 3-dimensional point from a unit edge reference cube. This was an interesting series of threes.

Explosion-implosion produced unit edged Platonic and Archimedean polyhedra from the tetrahedral, octahedral, and icosahedral families. They were produced continuously, in that, each point in the unit edged reference cube produced a different polyhedron. Points that were close to each other in the reference cube, produced polyhedron that were very similar in their shape.

Lalvani's *explosion-implosion* polyhedral transformation was conceptually clear from his dissertation and his exhibit. I was able to derive a model of this concept using C code to compute a model and produce computer graphics animation of the result, in **Figure 1**. The edge length preserving nature of this model is an ongoing point of fascination for me.

Acknowledgement

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References

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