Symmetric Linear Constructions in Motion

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Abstract

In this paper we show how to create sculptures which provide a sense of motion as arbitrary irregular polygons morph into planar stellar and planar convex affine regular polygons. For example, in Figure A, an irregular pentagon is transformed into a planar stellar pentagon which is morphed into a planar convex pentagon. With the exception of a few degenerate cases, affine regular polygons appear regular when viewed from a certain direction. Although a brass sculpture would be static, the morphing of one polygon into one or more other polygons provides the illusion of motion. We give precise instructions for creating such sculptures leaving only the variable of the initial irregular polygon as a complete unknown.

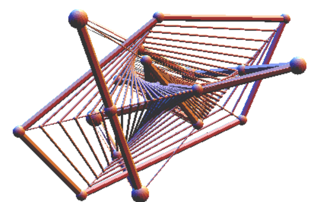

Figure A

0. Introduction to Symmetric Linear Constructions

In an abstract sense, symmetric linear constructions consist of a set of balls randomly suspended in fixed locations anywhere in 3-space to which we attach, sequentially, mathematically idealistic bungee cords. All bungee cords have a ball at each end. Many bungee cords also contain a third ball at various – but known – ratios from one end. These *mathematically idealistic* bungee cords can be stretched as long, or short, as we desire while remaining a taut straight line retaining the desired ratio. These bungee cords can only be attached to the sculpture by lining up any two balls on the bungee cord with two balls already on the sculpture. In our *mathematical world*, the two balls merge to become one. Furthermore, once we place a bungee cord on our sculpture it becomes rigid so that additional bungee cords will not distort previous bungee cords.

This creates the following puzzle: Knowing only the number, but not the placement of the initial balls, how do we select and how do we place the bungee cords so that the resulting sculpture will be pleasing and interesting?

We note in passing that the simplest "interesting" symmetric linear construction consists of connecting the midpoints of a quadrilateral. This is interesting because the resulting parallelogram is planar, moreover, the parallelogram can be turned in 3-space so that its shadow is a square. However, since this has little artistic value, we shall consider symmetric linear constructions with five or more balls.

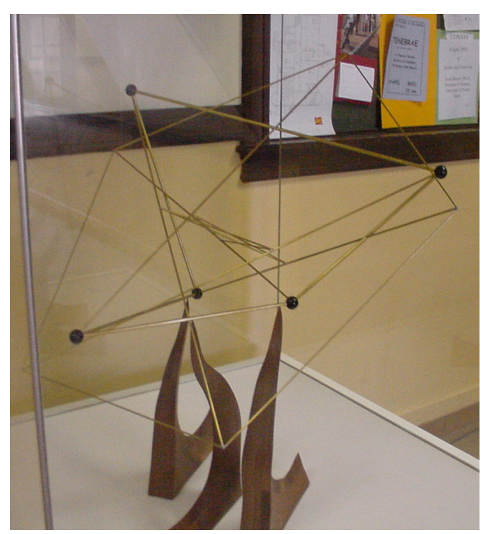

Figure 0.1

Figure 0.1, shows a pentagon solution to this puzzle. This sculpture, "Douglas Pentagon Theorem" by David A. Kaufman, is located at Bethel College in Newton, Kansas. In this paper, we provide instructions for building a variety of pentagon, hexagon, and heptagon sculptures. Each sculpture includes a sense of motion. All except the last three sculptures in this paper could be constructed in brass or wood. With the exception Bethel College's brass sculpture, shown in Figures 0.1 – 1.5, all sculptures in this paper are computer generated.

1. Static Pentagon Sculptures

We begin by constructing Bethel College's pentagon sculpture shown in Figure 0.1. First, we shall describe it using mathematical bungee cords and then we shall transform this construction into the real world of brass and wood.

Bethel College's Sculpture. Our first sculpture is made from 25 carefully selected bungee cords and five small balls labeled 1, 2, …, 5. In our *mathematical world,* we begin our sculpture by randomly suspending the five balls in fixed locations anywhere in 3-space. For the purpose of discussion, we shall color our mathematical bungee cords. For this sculpture we need: five brown bungee cords which have a ball on each end and a third ball located at the midpoint; five green bungee cords which have a ball at each end and a third ball located at a point $\sigma/(2+\sigma) \approx .447$ of the distance from one end where σ is the golden ratio $(1+\sqrt{5})/2$; five yellow bungee cords which have a ball at each end and a third ball located at a point $1/(2\sigma) \approx .309$ of the distance from one end; and ten red bungee cords which have a ball on each end. Note that the regular pentagon contains the golden ratio and, hence, its use here is not a surprise.

Constructing the Stellar Pentagon. We begin by attaching the ends of the five brown bungee cords to balls 1 and 2, to balls 2 and 3, to balls 3 and 4, to balls 4 and 5 and to balls 5 and 1. In our *mathematically idealistic world*, the initial balls do not move when the bungee cords are attached. At this stage we have constructed the highlighted portion of Figure 1.1, an irregular non-planar pentagon with the midpoints of each side constructed. (Note, not all midpoints appear centered due to perspective distortion.) Next, we attach the long end of a green bungee cord to ball 1 and the short end of the bungee cord to the midpoint of the opposite side of the pentagon and label the interior point I_1. The dotted lines in Figure 1.2 show two green bungee cords placed upon the sculpture. Likewise, we add the long end of each of the other four green bungee cords to balls 2, 3, 4, and 5 and we attach the short end of each cord to the midpoint of the side opposite the respective vertex. We label the interior points I_2, I_3, I_4, and I_5 respectively. In general, whenever we place a bungee cord on the construction, we note its position relative to ball 1 and then we place identical bungee cords on the construction at the corresponding positions relative to each of the other initial balls. This is the symmetry property of "symmetric linear constructions." The last step is to attach five red bungee cords to the five interior balls in order I_1 to I_2, I_2 to I_3, I_3 to I_4, I_4 to I_5, and I_5 to I_1. Jesse Douglas showed in [2] that the resulting pentagon will be an affine regular stellar pentagon regardless of the location of the original five points. Thus, the red pentagon will be planar, it will form a

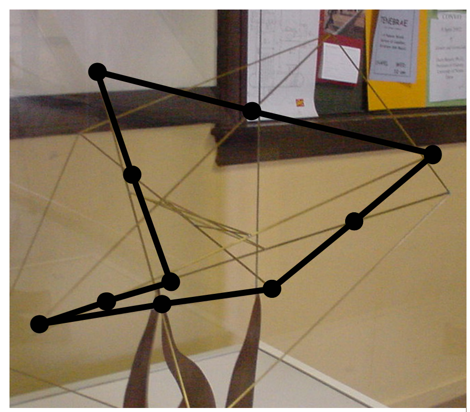

Figure 1.1

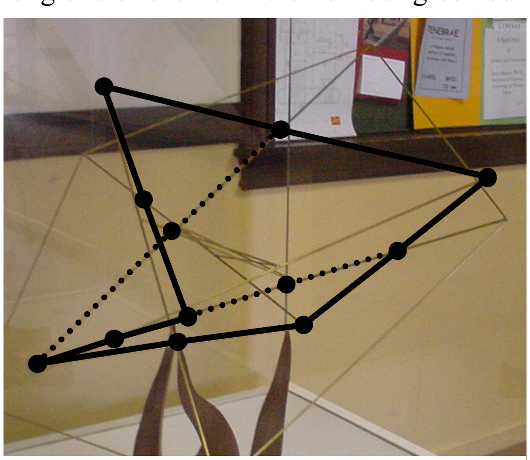

Figure 1.2

star and, usually*, the sculpture can be rotated so that the shadow of the red pentagon will form a regular stellar pentagon (assuming our light source is sufficiently far away). That is, all five lines will be the same length and the five angles at the points will be identical. It is not likely that the pentagon in the sculpture will be regular, and Bethel's is not. From a carefully selected viewpoint, we see in Figure 1.3 that the stellar pentagon looks approximately regular. (Errors are due to the placement of the camera and not to the sculpture.)

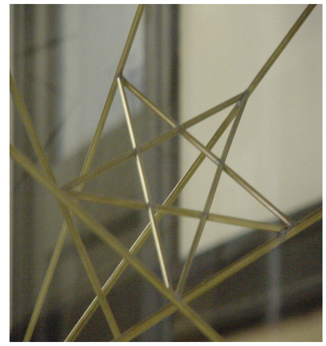

Figure 1.3

Constructing the Convex Pentagon. To finish Bethel's sculpture we must construct the convex planar pentagon. Ignoring the stellar pentagon just constructed, we start back at Figure 1.1. This time, we attach the long end of a yellow bungee cord to the original ball 1 and the *interior ball* of the yellow bungee cord to the midpoint of the opposite side of the pentagon. Notice that this time the bungee cord is not suspended between two points but rather extends beyond the side, which is not a problem in our mathematical world. The dotted lines in Figure 1.4 show two such bungee cords placed upon the Bethel Sculpture. Label the exterior ball E_1. Attach the other four yellow bungee cords in the corresponding fashion to the balls 2, 3, 4, and 5 and number the exterior balls of each cord E_2, E_3, E_4, and E_5 respectively. As before, attach five red bungee cords to the five exterior balls in order E_1 to E_2, E_2 to E_3, E_3 to E_4, E_4 to E_5, and E_5 to E_1. Jesse Douglas showed in [2] that the resulting pentagon will be an affine regular *non*-stellar pentagon. That is, regardless of the location of the original five points, the red pentagon will be planar, it will *not* form a star and, usually**, the sculpture can be rotated so that the shadow of the red pentagon will form a regular non-stellar pentagon (assuming our light source is sufficiently far away). It is not likely that the actual pentagon will be regular, and Bethel's is not, however, from a carefully selected viewpoint we see in Figure 1.5 that the non-stellar pentagon looks approximately regular.

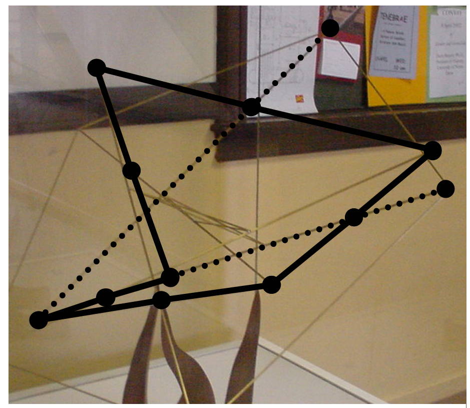

Figure 1.4

Figure 1.5

Constructions in Wood and Brass. In theory, the actual construction of this sculpture is relatively simple. Cut five rods of arbitrary length. Place balls at the midpoints of each rod. Join these five rods to form an arbitrary non-planar pentagon. Figure 1.1 shows and example of on such arbitrary pentagon. Label the balls consecutively. Merging the green and yellow bungee cords, as described above, forms a rod with four balls spaced as shown in Figure 1.6. Thus, one needs to measure the distance L from each vertex to the midpoint of the opposite side and then accurately cut a rod of

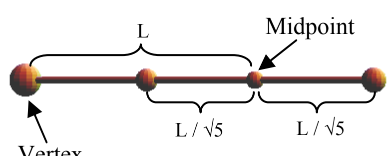

Figure 1.6

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* If the points are random, then with probability 1, this will happen. However, one can place the original points so that the five red balls are collinear. Moreover, if the original points form a planar regular convex pentagon then the five red balls will superimpose to form a single ball.

** As in the previous footnote, switching stellar and non-stellar.

length (1+1/√5) $L \approx 1.447\ L$. This rod needs to pass through the midpoint ball and into the opposite vertex. Place balls at a distance of (1/√5) $L = 44.7\ L$ on each side of the midpoint ball as shown on Figure 1.6. Label the interior and exterior balls the same number as the vertex. Finish by connecting the interior points in order and the exterior points in order. If all distances, measured on center, are accurate, then both the interior and the exterior pentagons will be planar and affine regular.

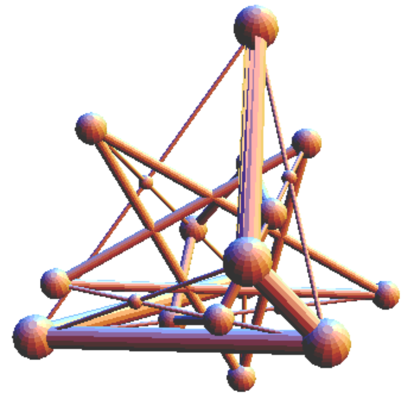

Figure 1.7

A Second Pentagon Construction. Instead of attaching the five brown bungee cords to consecutive balls, we attach them to every other ball to form secant lines. We use the same rods as shown in Figure 1.6, however, this time we place them so that L fits between a vertex and the midpoint of the secant line connecting the two balls adjacent to this vertex. As before, we connect the interior balls in order and the exterior balls in order. A computer generated sculpture of this construction is shown in Figure 1.7. The smallest balls are the midpoints of the secant lines. For this construction, the interior pentagon will always be convex and the exterior pentagon will always be stellar [1], which is the opposite of the previous construction.

Computer Constructions. All figures in this paper, with the exception of Figure 0.1 through Figure 1.4, were computer generated with Mathematica®. The power of Mathematica to allow variables to represent almost anything makes computer construction almost equivalent to construction steps for wood and brass sculptures. The initial balls, *p*[*i*], are randomly defined points in three space. Using these as starting points, we can then generate any of the desired points by applying the equation $a = (1\text{-}r)p+rq$. This equation creates a new point *a* located on line *pq* at the point *r* of ratio of the distance from *p* towards *q*. Note that if $r>1$, then the point *s* is located beyond *q*. For the initial pentagon construction, if we let *m*[1] be the midpoint of the opposite side, ½ p[3] + ½ p[4], and let *s*=1/√5, then the interior point is located at *i*[1]=*s p*[1] + (1 – *s*) *m*[1] and the exterior point is located at *e*[1]=(-*s*)*p*[1] + (1 + *s*) *m*[1]. The remaining step is to create surface graphics for the balls and for the cylinders between the balls. It is convenient to do this only once and define them ball[*point, size*] and rod[*point, point, size*]. Figure 1.6 can then be created by Show[ball[*p*[1],_], ball[*i*[1],_], ball[*m*[1],_], ball[*e*[1],_],rod[*p*[1],*e*[1],_] where each blank is filled in with a number to produce the desired size of each item. Notice, except for the definition of *p*[*i*], we never use coordinates. Adding in the other four rods, and connecting the original, interior and exterior pentagons, will produce a computer generated sculpture similar to Bethel College's sculpture shown in Figure 0.1. We note that adding the command <<RealTime3D`, including the << and the backwards quote, allows three dimensional objects to be spun in Mathematica with the mouse. Currently, this command is only partially functional and certain operations, such as colorings, may fail.

2. Pentagons in Motion

To add motion to the pentagon sculptures, we shall imagine the balls sliding. As shown in Figure 1.6, the original ball 1, interior ball I_1, exterior ball E_1, and the midpoint of the opposite side all lie on a line called a median line. Imagine five red balls, numbered 1 through 5, sliding down the five median lines starting from the vertices of the original pentagon. Figure 1.4 shows two such trajectories. These balls slide in unison starting from their respective vertex and slide at speeds proportional to the length of their rod so that they all reach the far end simultaneously. By connecting these five red balls in order with elastic red edges, we form a sliding red pentagon. Let $t = 1$ be the time it takes for each ball to slide from the vertex to the midpoint of the opposite side. Initially, at time $t = 0$, this sliding red pentagon is the original irregular pentagon. As time progresses the pentagon folds up and flattens out to become planar and stellar affine regular pentagon exactly at time $t = (1 - 1/\sqrt{5})$ when the five balls reach the critical interior location. As they pass this interior point, the sliding pentagon begins to unfold and, at time $t = 1$, forms an irregular non-planar pentagon with each ball located at the midpoint of the sides. The pentagon then flattens out again to form the planar convex affine regular pentagon at time $t = (1 + 1/\sqrt{5})$ when each ball reaches the critical exterior location. Unfortunately, such a sliding sculpture can only exist in our imagination and in computer animation. In the real world, the elastic red edges would get intertwined. Thus, we must consider sculptures which do not actually move, but do retain a sense of motion. We do this by including a collection of snapshots of this motion.

We add snapshots of this motion by adding a collection of extra balls to the rod in Figure 1.6. In Figure 2.1, thirteen extra balls have been added to each rod spaced at ratios L / 11. We connect these tiny extra balls together, in corresponding order, to create additional pentagons which show the snapshots of the sliding red pentagon. The sculptures shown in Figures 2.2 and 2.3 have 23 extra pentagons. In Figure 2.2, the irregular pentagon is morphed into the stellar planar pentagon and then is morphed into the convex planar pentagon. Note that each of the added pentagons is a symmetric linear construction, but none of these extra pentagons will be planar.

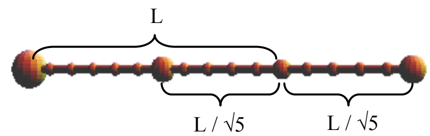

Figure 2.1

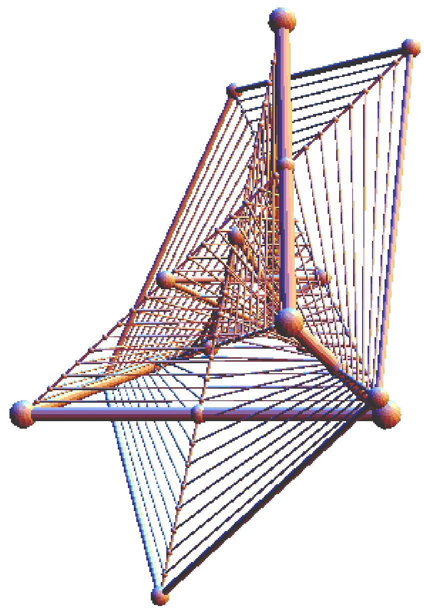
Figure 2.2

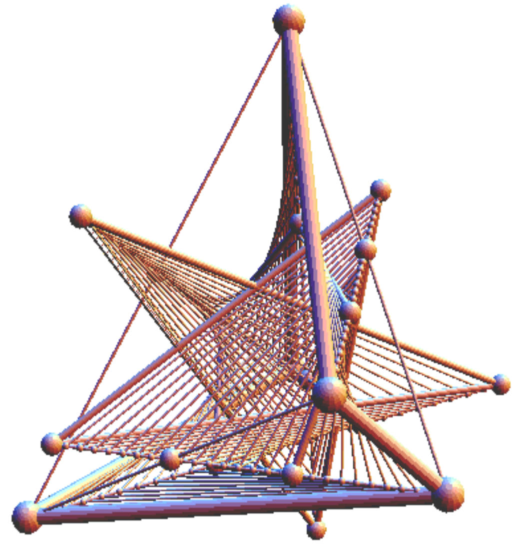
Figure 2.3

We can just as easily apply this idea to our second pentagon construction. This transforms the sculpture shown in Figure 1.7 into the sculpture shown in Figure 2.3. Notice, that in this sculpture, the irregular pentagon is first morphed into the convex planar pentagon and then morphed into the stellar planar pentagon. Also, notice that sculptures in Figure 1.7, Figure 2.2 and in Figure 2.3 begin with the same irregular pentagon.

An alternative to showing all of the morphing is to only morph from the stellar planar to the convex planar as shown in Figure 2.4.

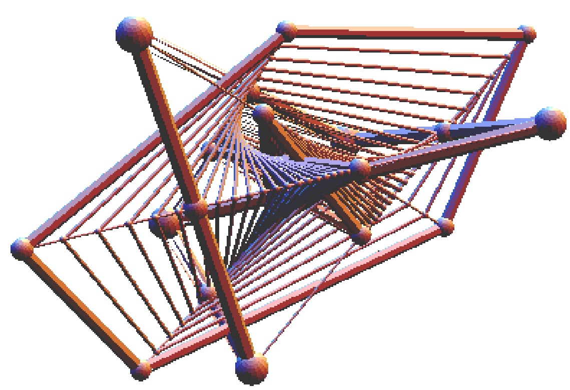
Figure 2.4

Translational Motion. To see the motion more clearly, we can slowly translate each image “vertically.” This will pull the constructed line segment off of the midpoint of the opposite side, but it will keep all lines straight. This produces a sculpture which starts from any irregular pentagon and morphs into a planar stellar pentagon and then unfolds into a planar convex pentagon as shown in Figure 2.5. Notice that each ball slides along a straight line at a constant but different speed. By using the pentagon construction that joins vertices to midpoints of the secant line we can start from the same irregular pentagon and morph it first into the convex planar and then fold it into the planar stellar pentagon as shown in Figure 2.6. Notice, it at first appears that it would be trivial to construct these sculptures backwards: start with any stellar and non-stellar pentagons placed as desired; connect with lines and morph one image into the next; continue the lines to form the correct ratios; connect to form an irregular pentagon. However, this sculpture would not be mathematically correct, since all of the midpoints must be translated by exactly the same distance in the same direction.

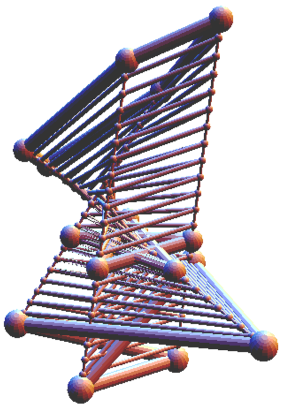

Figure 2.5

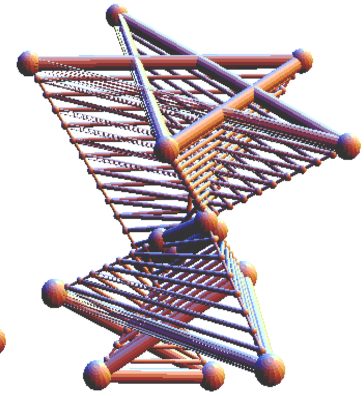

Figure 2.6

Construction Hints. To construct these out of wood or brass, one would need to locate the desired midpoints, either the midpoints of opposite sides or the midpoints of the secant lines. Next, translate each of these midpoints by attaching five identical parallel rods to these midpoints. Then construct five rods, as shown in Figure 2.1, to fit between each vertex and the translated midpoint. Finish, as above, by connecting corresponding balls to form pentagons. Remove the translation rods, if desired. The figures shown do not include the translational rods.

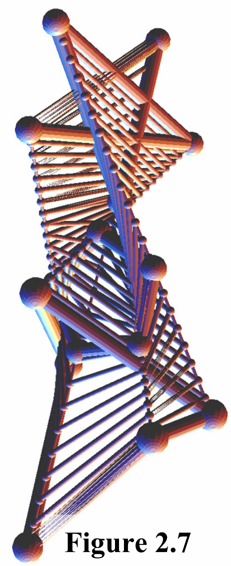

Figure 2.7

Note that this translation can be done in any direction. Figures 2.5 and 2.6 were translated in the direction of the average to the normal vectors to the planes of the two resulting planar pentagons. This keeps the planar pentagons as perpendicular as possible to the direction of translation. Figure 2.7 was translated along the unique line parallel to both planar pentagons. One can also choose to translate perpendicular to either one or the other of the planar pentagons. Or, one could choose a random direction to translate. Note that we are adding two steps not allowed under the pure symmetric linear constructions. Although not shown in the figures, we are adding the translational rods and we are making a decision about the direction of the translation after starting the sculpture. In the first section, no decisions were made once we started.

Notice that symmetric linear constructions preserves centers, which is the average of the vertices. Thus, all of the pentagons in the figures in the previous section had a common center and the centers of all pentagons in these figures follow along a translational line through the center of the sculpture.

3. Higher Order Polygons

In this section we touch briefly on a variety of sculptures of higher order. While there exist symmetric linear constructions which produce planar affine regular n-gons for every n, the construction requires the use of a minimum of n-3 bungee cords [1]. We show three such sculptures which could be built in brass, but then resort to sculptures which, most likely, can only be constructed via computer graphics.

Hexagon Construction. For hexagons, we start at a side, instead of a vertex, and place a midpoint on this side. We then locate the two balls one edge away from this side. Connecting these two balls forms a diagonal line. Find the midpoint of this diagonal line. Our critical ball is located one third of the distance from the midpoint of the side to the midpoint of the diagonal. Repeating this process for each side and connecting these critical balls forms a planar affine regular hexagon [1]. By adding snapshot balls, as shown in Figure 3.1, we obtain the sculpture shown in Figure 3.2. This sculpture shows an irregular hexagon, starting from the midpoints, morph into a planar affine regular hexagon. If we continue sliding these balls to the midpoints of the diagonals, we would fold the hexagon into a triangle.

Figure 3.1

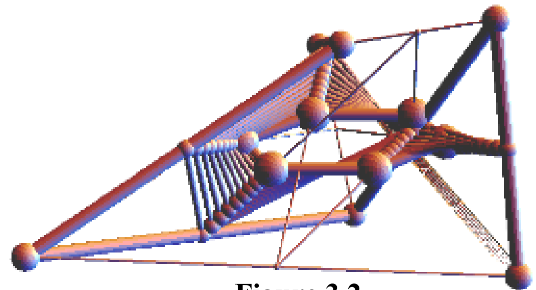
Figure 3.2

Heptagon Constructions. For a heptagon, we must construct four lines to identify the location of the critical red ball. Figure 3.3 shows one such construction containing all 28 lines required for constructing the planar convex affine regular heptagon. Figure 3.4 shows one set of four lines necessary for constructing one of the red balls for a different construction. This constructs a stellar affine regular heptagon. (For those curious, to construct the lines shown in Figure 3.4, begin by constructing the midpoint m_{24} between points p_2 and p_4 and the midpoint m_{15} between points p_1 and p_5. Then connect p_3 to m_{24} and place a yellow ball at a ratio of 1.29 from p_3 towards and past m_{24}. Connect this ball to m_{15} and place the red ball at a point .70 of the ratio from the yellow ball.)

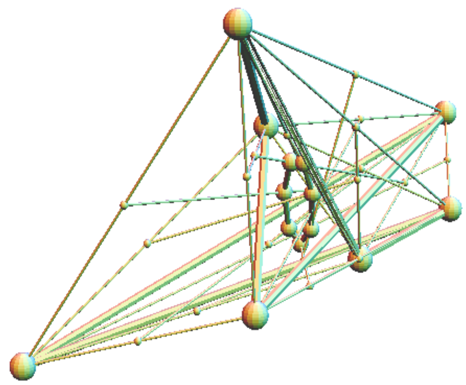
Figure 3.3

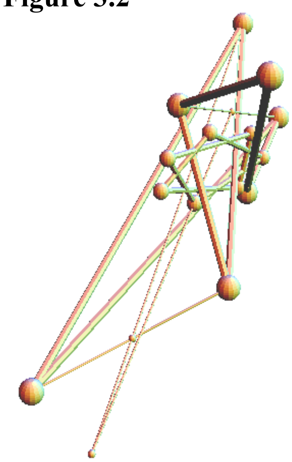
Figure 3.4

Weighted Averages. The easiest method to identify the desired location of the final position of the red balls is to calculate them as a weighted average of the initial balls. For example, the formula $(1/6)p_1+(1/3)p_2+(1/3)p_3+(1/6)p_4$ is a weighted average of four points, p_1, p_2, p_3, and p_4. To the computer, weighted averages are trivial to construct. For brass and wood, it is necessary to change weighed averages into symmetric linear constructions since locating a new ball along a line segment defined by two existing balls is constructible in brass. Algebraically, weighted averages can always be turned into symmetric linear constructions if desired. For example, the weighted average $(1/6)p_1+(1/3)p_2+(1/3)p_3+(1/6)p_4$ is equivalent to the hexagon construction stated above. And, in fact, weighted averages were used as a tool in determining the procedures described above [1]. The following sculptures were created using weighed averages, and therefore, the final polygons "hang in midair."

The Critical Weights and Type. One way to calculate a set of critical weights for an n-gon is the following. Let t be a positive integer less than $n/2$ and let $d_i = \cos(\pi / n) - \cos(t(2i+1)\pi / n)$ for each $i=1,2,\ldots,n$. Letting s be the sum of these distances and the critical weights are $w_i = d_i / s$. That is, for any random points $p_1, p_2, \ldots, p_n$ the points $q_k = w_1p_k + w_2p_{k+1} + \ldots + w_np_{k+n-1}$, with subscripts modulo n, will form a planar affine regular polygon when connected sequentially [1]. Moreover, the variable t describes the "type" of the polygon. If $t = 1$, then the resulting polygon will always be convex – each vertex will be connected to adjacent vertices. When $t = 2$, there will be a vertex between successively numbered vertices. The stellar pentagon is type 2 and so is the heptagon in Figure 3.4. In general, when $t = m$, the resulting polygon will have $m-1$ vertices between successively number vertices. Figure 3.7, below, shows the type *1*, type *2*, and type *3* heptagon. While it is most interesting when t is relatively prime to n, it need not be the case. When $t = 2$, for the hexagon, the resulting symmetric linear construction wraps the hexagon around the triangle formed by the midpoints of the diagonal twice. We see this in Figure 3.2 if we continue sliding the six balls up to the three diagonal lines.

Morphing Type. Once we fix the order of the polygon, the only variable in our formula for weighted averaged is the type *t*. Thus we now morph one image into the next by letting the *t* change. Only when type is a whole numbers will the polygon be planar. We can not let *t=0*, and so we let *t* slide from one natural number to another.

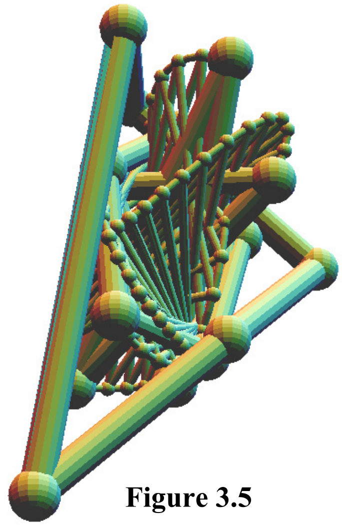

Figure 3.5

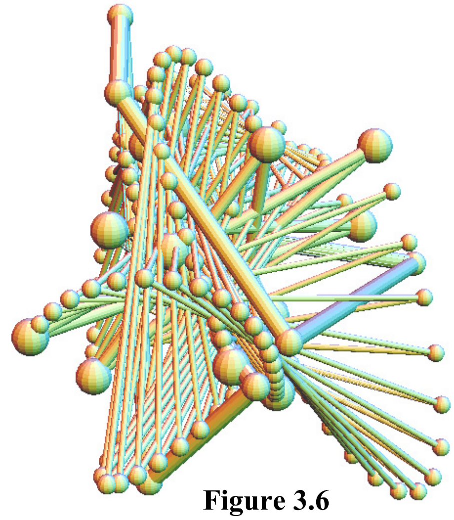

Figure 3.6

Figure 3.5 shows an irregular pentagon with the convex pentagon morphing into the stellar pentagon by morphing type. Within this sculpture are five beads of string connecting each vertex of the stellar affine regular pentagon to its corresponding vertex on the convex affine regular pentagon. Although nothing connects to the original polygon, the sculpture is completely defined by the location of the initial random pentagon. Figure 3.6 consist of an irregular heptagon and the type 2 planar heptagon morphing into a type 3 planar heptagon.

Morphing Type with Translational Motion. Adding in translations to stretch out this motion – and by removing the irregular *n*-gon we get the Figure 3.7 which morphs an affine regular convex heptagon into a type 2 stellar affine regular heptagon and then into a type 3 affine regular stellar heptagon.

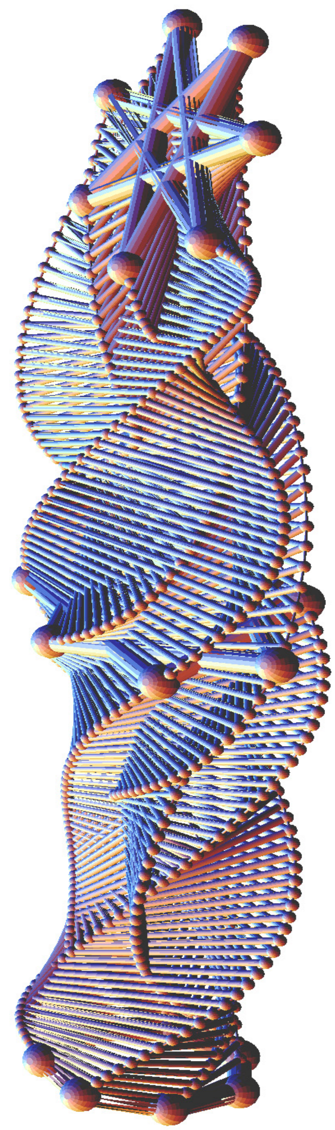

Figure 3.7

Summary

Symmetric linear constructions and the weighted averages provide a method for creating a variety of predictable and pleasing sculptures which begin from irregular polygons. By morphing these images from either the random polygon to a planar affine regular, or by morphing these images from one planar type to one or more other types, one produces unexpected curves which fold and unfold the polygon.

References

[1] Douglas G. Burkholder, *Parallelogons and Weighted Averages*, Revista de Física y Matemática (FISMAT) de la Facultad de Ciencias, Escuela Poltécnica Nacional (Quito-Ecuador), XII, to Appear.

[2] Jesse Douglas, *A Theorem on Skew Pentagons,* Scripta Mathematica, Vol. 25, 1960.