Dynamics on Discrete Structures: A Dialog between Squares and Circles

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Discretized versions of continuous structures can be used to great effect to increase the amount of information conveyed in mathematics, art and science. As an example, we will examine a model for a confined polymer in a solution, Figure 1, as imagined by Pierre-Gilles de Gennes [1], and an analogous mathematical model of the same decomposition of space, rotated counterclockwise through 90 degrees as in Figure 2, [3].

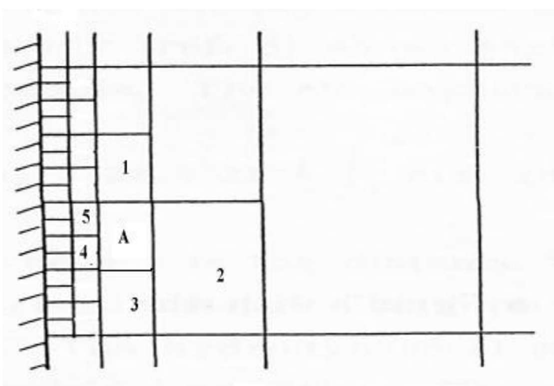

Figure 1: *Absorbed polymer layer*

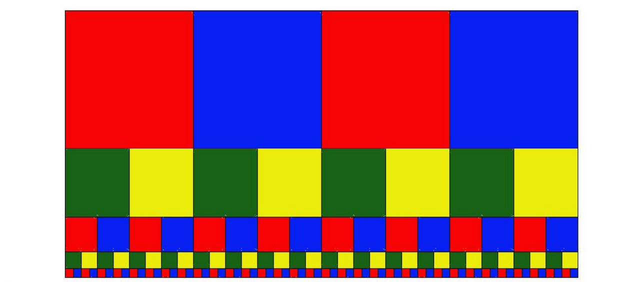

Figure 2: *Hyperbolic tiling of half-space*

To visualize the theory of Stochastic Schramm-Loewner Evolutions [5] in the upper-half space, where a continuous set of conformal maps is generated by applying at the boundary a random process, traditionally one looks at the trace of the set of points that are eliminated in time. Unfortunately, this visualization conveys very little information regarding the family of maps. The hyperbolic tiling in Figures 1 and 2 can be taken as a discrete approximation of the upper-half space on which to apply the dynamics just described. While these rectangular models serve the purpose of performing a hyperbolic tiling of the upper-half plane, the fundamental rectangular blocks are interlocked and this rigidity does not allow for natural dynamics to occur.

To move things about, we notice that the essential component in the tiling of both Figure 1 and Figure 2 is their combinatorial structures, among other things the fact that every square has exactly five neighbors, thirteen square which are two away and so on. One can transfer this structure to the unit circle by using the theory of circle packing, recently developed in complex analysis [4]. A basic result of the theory is that any finite, connected planer graph gives rise to a unique packing, modulo some basic transformations [2].

To build a circle packing isomorphic to the hyperbolic tiling of Figure 2, we could construct a graph by connecting the centers of adjacent rectangles in the composition. The associated circle packing is a moveable configuration where every circle has five neighbors, thirteen away and so forth, that is the combinatorics of the original tiling is preserved. A portion of this packing with circles away from the boundary is shown in Figure 3, which we generated using the computer algebra system Maple. Each circle and its neighbors represents a rectangle of the planar decomposition of Figure 2 and its adjacent rectangles. The radius of the circle is proportional to the distance from the boundary.

A dynamical system modeling the effect of applying a random variable on the boundary can be introduced on the circle packing by randomly deleting circles on the boundary, which corresponds to deleting vertices from the original graph, and then repacking the unit circle with the new graph. One can then visually retrace the entire history of the random walk on the boundary by examining the fractal like set of circles from the original packing, which have been deleted in time. The construction gives a clear visualization of Schramm-Loewner Evolutions.

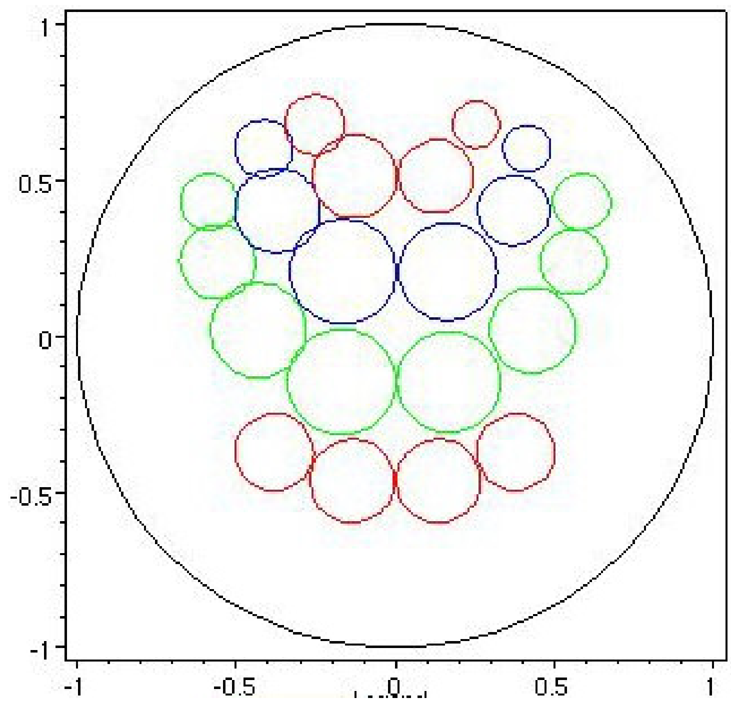

Figure 3: *Circle packing*

References

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[5] W. Werner, *Random Planar Curves and Schramm-Loewner Evolutions,* Lecture Notes, Saint-Flour, 2002.