An Approach in Coloring Semi-Regular Tilings on the Hyperbolic Plane

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Abstract

A coloring of a semi-regular tiling is perfect if every symmetry of the tiling permutes the colors of the tiling. In this paper, an approach to the construction of perfect colorings of semi-regular tilings on the hyperbolic plane is presented.

1. Introduction

In [3], a method for coloring symmetrical patterns was presented where a fundamental domain of the pattern is assigned exactly one color. In this paper, we present a general framework for coloring planar patterns where a fundamental domain of the pattern may be assigned more than one color. We apply the framework to construct perfect colorings of semi-regular tilings on the hyperbolic plane. We will use the subgroup structure of the symmetry group of the tiling to systematically construct the colorings.

An **edge-to-edge tiling** is a plane tiling where the corners and sides of the polygonal tiles form all the vertices and edges of the tiling and vice versa. A vertex of an edge-to-edge tiling is said to be of **type** $p_1 \cdot p_2 \cdot \ldots \cdot p_q$ if the polygons about this vertex in cyclic order are a p_1-gon, a p_2-gon, …, and a p_q-gon. An edge-to-edge tiling having regular polygons as its tiles with vertices all of the same type, and where the symmetries of the tiling act transitively on the vertices is called **semi-regular**. We denote the semi-regular tiling as $p_1 \cdot p_2 \cdot \ldots \cdot p_q$ depending on its vertex type $p_1 \cdot p_2 \cdot \ldots \cdot p_q$. If the polygons in the tiling are of the same type, particularly a *p*-gon meeting *q* at a vertex, we denote the tiling as p^q.

In this paper, we will present an approach to color perfectly the $8 \cdot 8 \cdot 5$ and $4 \cdot 10 \cdot 8$ hyperbolic semi-regular tilings.

2. General Framework for Coloring Planar Patterns

The following general framework for coloring planar patterns shall be used to obtain colorings of semi-regular tilings.

Let X be the set of tiles in the tiling to be assigned colors;
G be the symmetry group of the uncolored tiling;
H be the subgroup of elements of G permuting the colors;
C be the set of colors.

Let O_i $(i \in I)$ be the *H*-orbits of colors and c_i a color in O_i. Then $O_i = \{hc_i : h \in H\}$ and corresponding to this set is the set $\{hJ_i X_i : h \in H\}$, where J_i is the stabilizer in *H* of the color c_i and X_i consists of representatives of each *H*-orbit of elements of *X* with the representatives colored c_i.

The following are true:

1. The action of *H* on O_i is equivalent on its action on $\{hJ_i : i \in I\}$ by left multiplication.
2. In O_i, the number of colors is $[H : J_i]$.
3. If $x \in X_i$ then $Stab_H(x) \leq J_i$, where $Stab_H(x) = \{h \in H : hx = x\}$.
4. If $x \in X_i$ then $|Hx| = [H : J_i] \cdot [J_i : Stab_H(x)]$
5. The number of *H*-orbits of colors is less than or equal to the number of *H*-orbits of elements of *X*.

The following steps, based on the general framework, shall be used to obtain the required coloring of the tiling.

1. Determine the finite group *S* of isometries in *G* that stabilizes a representative tile *t* from an orbit.
2. Determine all subgroups *J* of *G* such that $S < J$.
3. If tile *t* has color *c*, apply *c* to all tiles in the set *Jt*. This makes *J* the stabilizer of the color *c* inside *G*. If $[G : J] = k$, then *Jt* is $\frac{1}{k}$ of the tiles in the class where *t* belongs.
4. To complete the coloring, assign a color to every element of the set $\{gJt : g \in G\}$. One element of this set has color *c*, which is *Jt*. There should be $k-1$ other elements or colors.

Hence, the index *k* of the subgroup *J* in *G* is the number of colors that can be used to perfectly color the orbit of tiles containing *t*.

3. Coloring the Hyperbolic Plane

3.1. Tessellating the Hyperbolic Plane. In [1], Aziz created a computer program Coloring the Hyperbolic Plane (CHP) that tessellates the hyperbolic plane with congruent triangles of interior angles $\frac{\pi}{p}$, $\frac{\pi}{q}$, and $\frac{\pi}{r}$, where $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} < \pi$. Denote by *e* one of the triangles of the tessellation and call it the fundamental triangle. Let *P* be the reflection on the side of the triangle opposite the angle $\frac{\pi}{p}$, *Q* as the reflection on the side of the triangle opposite the angle $\frac{\pi}{q}$, and *R* the reflection on the side of the triangle opposite $\frac{\pi}{r}$. The symmetry group *G* of the tessellation is generated by *P*, *Q*, and *R*, denoted by $*pqr$.

Given the fundamental triangle *e*, the tessellation may be recovered by getting the images of *e* under *P*, *Q*, *R*, and their products. There is a one-to-one correspondence between the elements of *G* and the triangles in the tessellation. Each triangle in the tessellation can then be labeled by the corresponding element of *G*. The action of *G* on the triangles of the tessellation, where $g \in G$ acts on a triangle by sending it to its image, is equivalent to the action of *G* on itself by left multiplication.

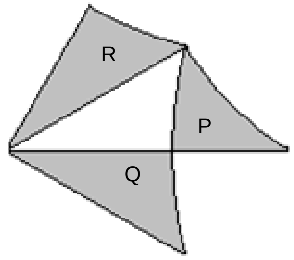
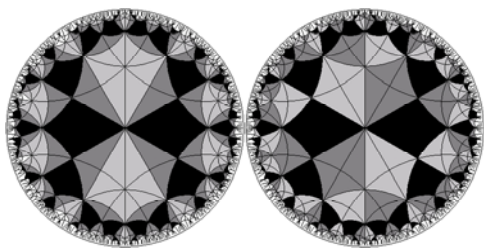


Figure: (1) Labeling the triangles in the tessellation

3.2. Coloring Using Right Cosets. If *S* is a subgroup of *G* of index *n*, a coloring using right (or left) cosets of *S* refers to a bijective map from the set of right (or left) cosets of *S* to a set of *n* colors. Triangles labeled by elements of a right (or left) coset are colored using the color assigned to the coset. In Figure 2, we give a coloring of the hyperbolic plane using right cosets of the subgroup *S* and in Figure 3, we give another coloring using left cosets of the subgroup *S*, where *S* represents a subgroup of index 3 of the hyperbolic triangle group $*642$.



2 3

Figures: (2) Right coset coloring using *S*; (3) Left coset coloring using *S*

The right coset colorings of a given subgroup *S* of the symmetry group *G* of the tessellation plays an important role in studying the subgroup structure of *G*. *S* turns out to be the symmetry group of the colored tessellation and *S* fixes the colors of the tessellation. In this paper, we will use the right coset colorings generated by CHP to determine the subgroups of *G* that contain the stabilizer of the tiles in the given semi-regular tilings.

4. Perfect Colorings of Semi-Regular $8 \cdot 8 \cdot 5$ and $4 \cdot 10 \cdot 8$ Tilings

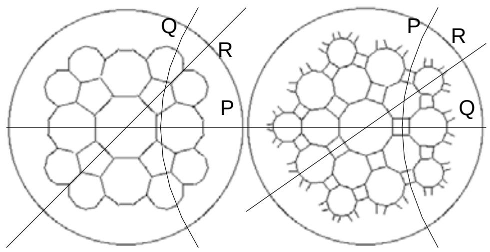
In this part of the paper, we illustrate how to obtain perfect colorings of semi-regular $8 \cdot 8 \cdot 5$ and $4 \cdot 10 \cdot 8$ tilings using the given framework. Both tilings have symmetry group $G = *542$; *G* contains rotations of order 5, 4, 2 with centers of the corresponding rotations lying on mirror lines.

In coloring the semi-regular tilings, we will make use of the subgroups of *G*. GAP [8] is used to generate a listing of the subgroups of *G* shown in Table 1. For the purposes of this paper, and due to coloring constraints, we will only consider subgroups up to index 5.

List of Subgroups of *542 of Index <= 5: Number of Subgroups = 7
1 Group([Q, R, P])
2 Group([Q, R, PRP])
2 Group([RQ, P])
2 Group([RQ, PQ])
4 Group([RQ, PRPQ])
5 Group([Q, P, RPR, RQRPRQR])
5 Group([Q, P, RPQR])

Table: (1) Subgroups of $*542$ of index less than or equal to 5

The generators *Q*, *R*, *P* appearing in Table 1 are mirror reflections with axes shown in Figures 4 and 5 for the respective tilings $8 \cdot 8 \cdot 5$ and $4 \cdot 10 \cdot 8$.



4 5

Figures: (4-5) Generators *Q*, *R*, and *P*

4.1. Semi-Regular $8 \cdot 8 \cdot 5$ Tiling. The semi-regular $8 \cdot 8 \cdot 5$ tiling has two orbits of tiles: the orbit of 8-gons and the orbit of 5-gons. To construct the perfect colorings, we color each orbit of tiles separately. We first color the orbit of 8-gons.

First, note that the finite group that stabilizes an 8-gon is of type D_4, the dihedral group of order 8. We need to select the subgroups J_i that contains D_4. The condition that J_i contains the stabilizer is always satisfied by *G*. To find other subgroups containing D_4, we will use the right coset colorings of the subgroups of *G*. To obtain the right coset colorings, we use the CHP program.

From the program, Figure 6 shows the right coset coloring using the subgroup $J_1 = \langle Q, P, RPR, RQRPRQR \rangle$. Note that the subgroup *D* generated by the 90° rotation about the indicated point *x* and mirror about the horizontal line through *x* fixes the color of the given right coset coloring. Thus, the subgroup $J_1 = \langle Q, P, RPR, RQRPRQR \rangle$ contains the group of type D_4 and can now be used to color the orbit of 8-gons for the $8 \cdot 8 \cdot 5$ tiling.

Figure 4 shows the right coset coloring using the subgroup $J_2 = \langle Q, P, RPQR \rangle$. Similarly, the subgroup *D* generated by the 90° rotation about *x'* and mirror about the horizontal line through *x'* fixes the colors of the coloring. Thus, the subgroup J_2 also contains D_4.

We are now ready to color the orbit of 8-gons. We will use the subgroups J_i, namely J_1, J_2, and $J_3 = G$. Using $J_3 = G$, we color all 8-gons using one color to obtain the coloring in Figure 8.

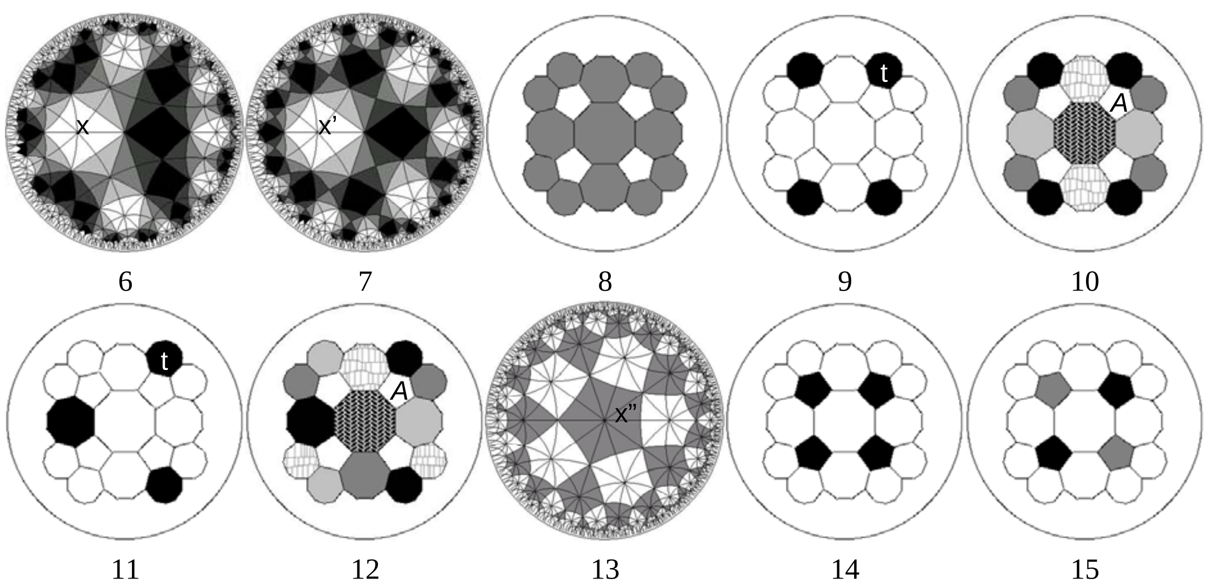
Next, we color the orbit of 8-gons using $J_1 = \langle Q, P, RPR, RQRPRQR \rangle$. To obtain a perfect coloring using J_1, we first choose a representative tile *t* from the 8-gons. We then color $J_1 t$ with black, as seen in Figure 9. To color the rest of the orbit, we apply the 5-fold rotation with center *A* lying on mirrors *R* and *Q* on $J_1 t$ to obtain a coloring of five colors given in Figure 10.

Lastly, we color the orbit of 8-gons using $J_2 = \langle Q, P, RPQR \rangle$. Coloring all tiles in $J_2 t$ black, we obtain Figure 11. Then we assign 4 different shades and textures of gray to the tiles in the other orbits by applying the 5-fold rotation about *A* to obtain Figure 12.

Next, we color the orbit of 5-gons. The finite group that stabilizes a 5-gon is of type D_5, the dihedral group of order 10. We now select the subgroup $J_{i'}$ that contains the stabilizer. Aside from *G*, our choice for $J_{i'}$ is the subgroup $\langle Q, R, PRP \rangle$. Figure 13 shows the right coset coloring using $\langle Q, R, PRP \rangle$. The subgroup generated by 72° about *x*" and mirror reflection about the horizontal line through *x*" fixes the colors of the coloring and is of type D_5.

To color the orbit of 5-gons, we let $J_{1'} = G$ to obtain Figure 14 and let $J_{2'} = \langle Q, R, PRP \rangle$ to obtain Figure 15.

To color the entire semi-regular tiling, we combine all the colorings of each orbit of tiles above. Thus, the resulting perfect colorings of the $8 \cdot 8 \cdot 5$ tiling are shown in Figure 16.



Figures: (6-7) Right coset colorings of J_1 and J_2 respectively; (8) Perfect coloring of the orbit of 8-gons using $J_3 = G$; (9) $J_1 t$; (10) Perfect coloring of the orbit of 8-gons using J_1; (11) $J_2 t$; (12) Perfect coloring of the orbit of 8-gons using J_2; (13) Right coset colorings of $J_{2'}$; (14-15) Perfect coloring of the orbit of 5-gons

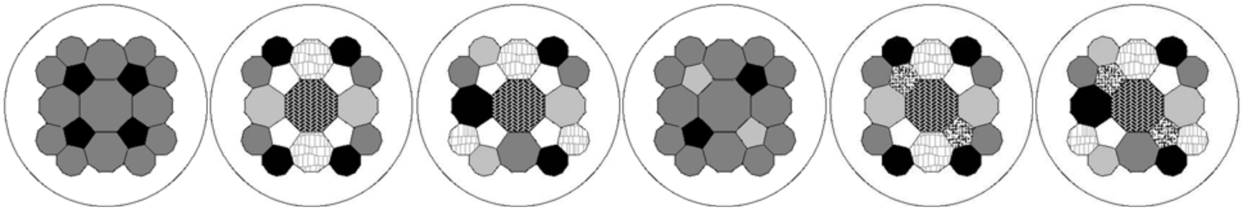


Figure: (16) Perfect colorings of the $8 \cdot 8 \cdot 5$ tiling

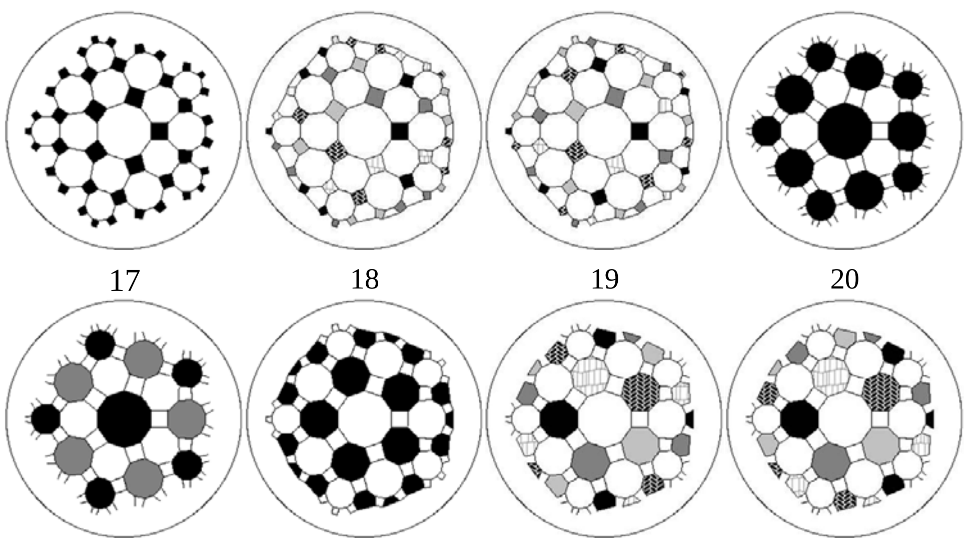
4.2. Semi-Regular $4 \cdot 10 \cdot 8$ Tiling. The semi-regular $4 \cdot 10 \cdot 8$ tiling has three orbits of tiles: the orbit of 4-gons, 10-gons, and 8-gons. We follow the steps given in 4.1 and color each orbit of tiles separately.

To color the 4-gons, we use $J_1 = G$, $J_2 = \langle Q, P, RPR, RQRPRQR \rangle$, and $J_3 = \langle Q, P, RPQR \rangle$, where $D_4 < J_i$. We have the three colorings in Figures 17, 18, and 19.

Next, we use $J_{1'} = G$ and $J_{2'} = \langle Q, R, PRP \rangle$, where $D_5 < J_{i'}$, to color the 10-gons shown in Figures 20 and 21.

Lastly, we use $J_{1''} = G$, $J_{2''} = \langle Q, P, RPR, RQRPRQR \rangle$, and $J_{3''} = \langle Q, P, RPQR \rangle$, where $D_2 < J_{i''}$, to color the 8-gons in Figures 22, 23, and 24.

Next, we combine all these colorings to obtain the perfect colorings of the $4 \cdot 10 \cdot 8$ semi-regular tiling as seen in Figure 25.



17 18 19 20

21 22 23 24

Figures: (17-19) Perfect coloring of the orbit of 4-gons; (20-21) Perfect coloring of the orbit of 10-gons; (22-24) Perfect coloring of the orbit of 8-gons

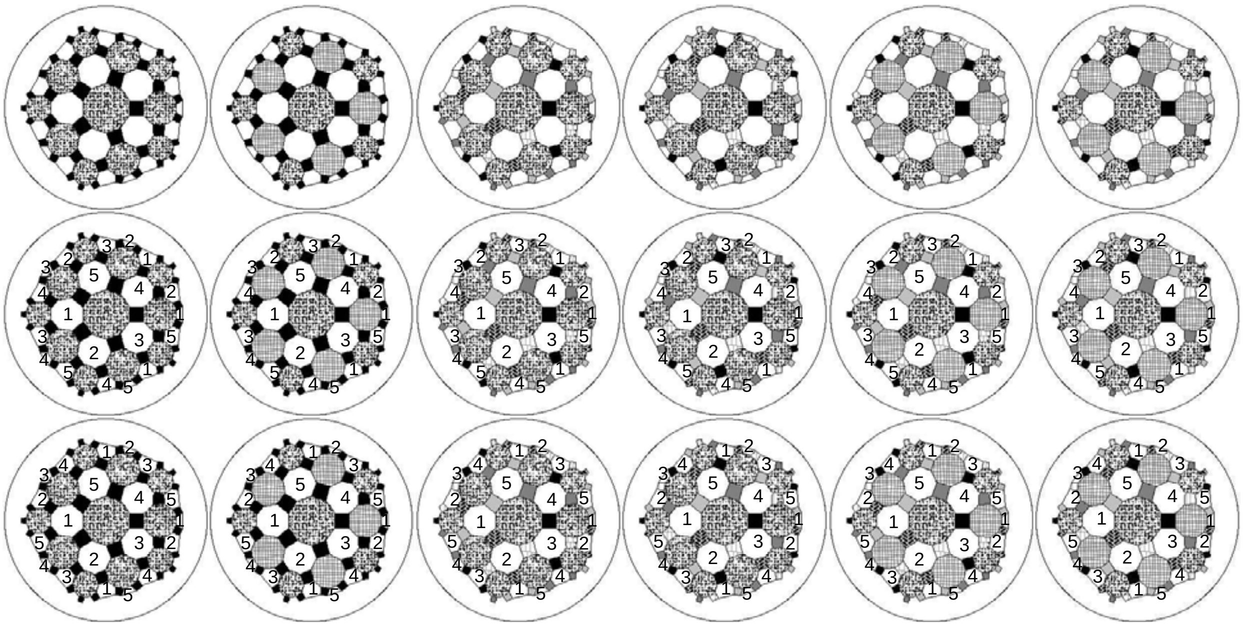
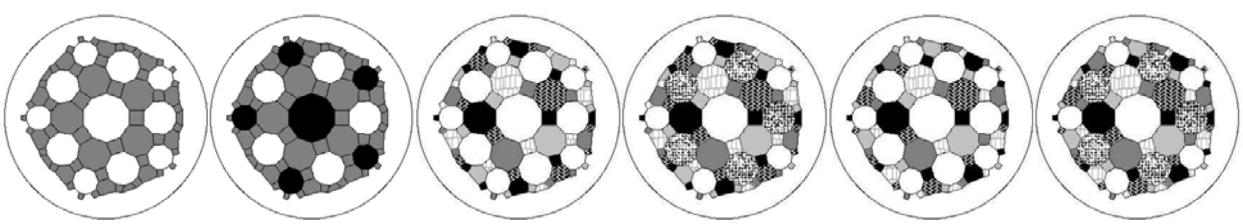


Figure: (25) Perfect colorings of the $4 \cdot 10 \cdot 8$ tiling where the orbits do not share colors

Observe that if J_i is used to color one orbit of tiles, it can also be used to color a second orbit of tiles as long as J_i contains the stabilizer of a tile in the second orbit of tiles. Moreover, if a color used to color tile *t* in the first orbit of tiles is to be used to color tiles in the second orbit, then the tile *t'* that will be assigned the same color as tile *t* should have a stabilizer contained in J_i.

In coloring the $4 \cdot 10 \cdot 8$ tiling, the orbit of 4-gons and the orbit of 8-gons can share the same color. These colorings appear in Figure 26. The colorings A and B are obtained using $J_1 = G$ to color both orbits of 4-gons and 8-gons. The colorings in C and D are obtained using $J_2 = \langle Q, P, RPR, RQRPRQR \rangle$ while the colorings in E and F are obtained using $J_3 = \langle Q, P, RPQR \rangle$.



A B C D E F

Figure: (26) Perfect colorings of the $4 \cdot 10 \cdot 8$ tiling where the orbits share colors

5. Conclusion

In this note, we give an approach to color semi-regular tilings on the hyperbolic plane. We use the general framework for coloring planar patterns where an orbit of tiles in the given tiling is colored using a subgroup of the symmetry group *G* of the tiling containing the stabilizer of the tile. We use the GAP program to generate the subgroups of *G* while a helpful tool in studying more closely the subgroup structure of *G* is the CHP program.

We intend that the approach provided here in obtaining perfect colorings of semi-regular tilings will provide a springboard in the construction of colorings (both perfect and non-perfect) of tilings in general on the hyperbolic plane.

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