A Perspective on Infinity: Anamorphism and Stereographic Projection

Michael Frantz
Mathematics/Physics/Computer Science Department
University of La Verne
1950 Third Street
La Verne, CA, 91750, USA
E-mail: frantzm@ulv.edu

Abstract

This paper introduces the concept of anamorphism in art and describes the various forms it has taken throughout history. Examples of direct and catoptric anamorphisms are presented, and geometrical methods for anamorphic processes are briefly sketched. An example of a current work of anamorphic art is discussed, and more utilitarian applications briefly summarized. The concept of the stereographic projection between the complex plane and the Riemann sphere is introduced, and various properties of potential interest to artists described. The role that stereographic projection might play in the expression of the infinite is explored.

1. A Brief Introduction to Anamorphism

Although there are few persons who would admit to a love for anamorphic art, or even be able to reasonably express the meaning of the term, the fact remains that virtually everyone in America has been exposed to it and that most are patrons on a quite regular basis. Each week millions of Americans attend the cinema or rent movies, and in the process unwittingly become consumers of an artistic technique that had its roots in the fifteenth century – but more on that later.

The word *anamorphosis* comes from the Greek ana (again) and morphe (form), defined in the unabridged second edition of Webster as: "a drawing presenting a distorted image that appears in natural form under certain conditions, such as when viewed at a raking angle or reflected from a curved mirror". The technique was devised by Renaissance-era artists who were beginning to explore a new concept called visual perspective, which was transforming the world of art by providing artists the ability and control to shape the borders between reality and perception. It is generally accepted that the earliest known example of an anamorphism is a child's face (Fig. 1), drawn by Leonardo da Vinci in 1485, and appearing in his Codex Atlanticus [1]. The image on the right is the result of a viewpoint off to either side of the elongated facial image, and subtending an extremely acute angle with the picture plane. Not surprisingly, sketches of Piero della Francesca indicate that he at least knew of the principles involved even before da Vinci, even if he did not put them into practice in any example that survived until today.

Fig. 1: *Head of Child sketch, Leonardo Da Vinci (1485); original and from skewed viewpoint*

Once the techniques became more widely known among artists and they had undergone refinements, anamorphic images became tools in the 16^{th} and 17^{th} centuries for exploring politics and religion. Artists took advantage of the methods to safely produce controversial (and even heretical) visual statements for public viewing, by camouflaging the real imagery behind a mask of gross perspectival distortion. Of course, the original visual data could be recovered by simply viewing the work from the correct position, which may or may not have been easily ascertainable by the general public.

The popularity of anamorphism waned after 1850, and in the late 19th and early 20th centuries, these drawing techniques were mostly regarded as visual games, puzzles, and curiosities of bizarre perspective. The concealing effects of anamorphic perspective also served as an ideal vehicle for the concealed public display of erotic images, including a series by Salvador Dali and numerous erotic series from Asia. It took until much later in the 20th century for the entertainment industry to discover one of the most important practical applications of descendants of Leonardo's original technique in film projection. Many other applications of the method are in current use, by no means limited to the world of fine art or the entertainment industry.

2. Direct (Oblique) Anamorphosis

The direct, or oblique anamorphosis was the first anamorphic form, and consisted of the rendering of a normal scene in such a way that one could only recover the original image by placing the eye almost on the picture plane itself, and off to one side. The extreme foreshortening then recovered the original image, which when viewed straight on, would often be an unintelligible smear of elongated forms. Several famous examples serve to illustrate the technique. Erhard Schön, a German printmaker and student of Albrecht Dürer, produced his Vexierbild (Puzzle Picture) in 1535, portraying Emperor Charles V, Ferdinand I of Austria, Pope Clement VII, and Francis I (Fig. 2). If one views the painting obliquely at a very acute angle from the left or right, the four visages become apparent, as well as German and Latin inscriptions which reveal their identities. Within the background of each of the four layers is a scene depicting the events connected with each leader: a military scene, the siege of Vienna, God threatening a Turk, and Orientals and a camel. [2]

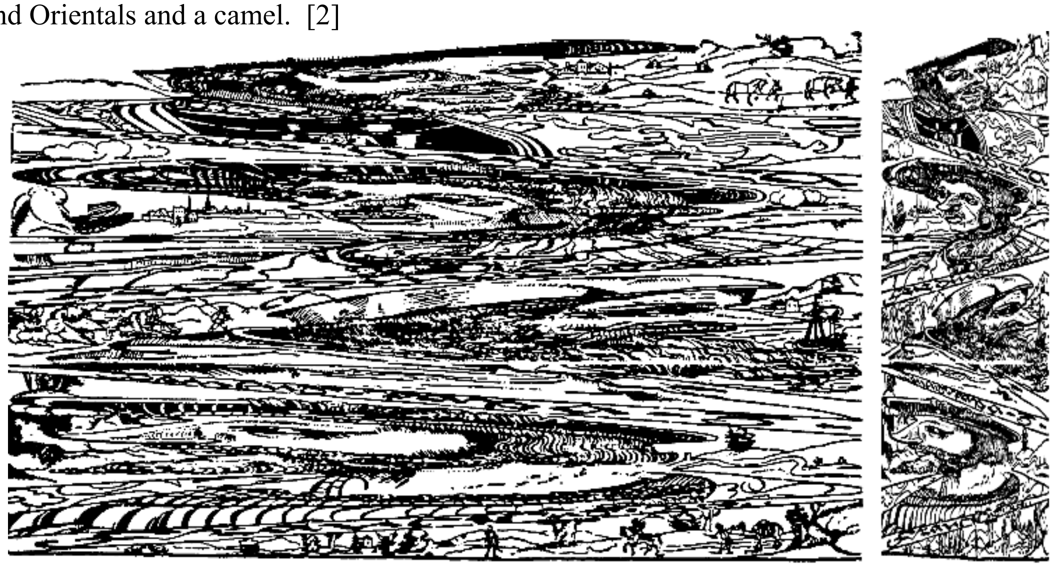

Fig. 2: *Vexierbild, Erhard Schön (1535); straight on view and oblique view*

Perhaps the most well-known anamorphic artwork is the 1533 painting by Hans Holbein titled *The Ambassadors* (Fig. 3a). If mounted on a staircase, then a patron looking up or down at the painting as he wandered up or down the stairs might have seen the strange object in the foreground transform into a skull (Fig. 3b). Jim Hunt, a retired physics professor at the University of Guelph, points out that the artist's name derives from *hohle Bein*, meaning hollow bone, and the image imparts an ironic twist that serves as "a reminder of the vanity of all earthly things. It was a moral tale." [3]

Figure 3a: *The Ambassadors, Hans Holbein (1533)*

Figure 3b: *skull from "correct" viewpoint*

3. Direct and Catoptric Cylindrical and Conical Anamorphoses

The early to mid-17th century saw the introduction of new types of anamorphoses utilizing drawing and painting directly on the surfaces of cones, and pyramids with bases of four, five or six sides, or onto a planar surface which could then be folded into one of these objects. If the object stood on its base, then the unique viewpoint yielding the intended final image was at a point above the vertex of the cone or pyramid. Naturally, the most distortion occurred near the center of the image corresponding to the tip of the vertex of the solid. Jean du Breuil, a French scholar of the mid-17th century, filled rooms with these objects, jutting out from walls, hanging from ceilings, and resting on tables, as illustrated in [4].

An important variation on this method also appeared at roughly the same time which involved the use of a mirrored cone. Anamorphoses depending on a mirror for reconstruction of an image are referred to as *catoptric* anamorphoses, from the Greek *katoptron* for mirror, literally *something that looks back*. In this case, the image which one expects to see from the vertex viewpoint must be painted or drawn on the flat surface on which the base of the cone rests, surrounding the cone in a circular image with a central disk missing (for the cone). Since most cones had rather steep slant edges, the distortion apparent in the planar image was of a radical nature, and the artist also had to contend with the fact that such a reflected perspective must undergo an inversion process, so that small areas of the image near the vertex of the cone must translate into large regions smeared all around the outer circumference of the circular planar source work. Figure 4a shows a top down view of such a conical source image of a rider on his horse, while the restored image appears once the conical mirror is in place (Fig. 4b). This particular image appeared in the "Devices of Wonder" exhibit at the Los Angeles Getty Museum in 2002 [5].

Fig. 4a: *Conical source image without mirror*

Fig. 4b: *Image restored by conical mirror.*

Catoptric anamorphoses more commonly utilized a mirrored cylinder rather than a cone. In this case, the source planar image would also have a central blank disk to mark the intended position of the cylinder, and since the painting to be reflected would often be on a rectangular canvas, this caused curious distortions in the cylindrically reflected image as the rectilinear boundaries of the frame were transformed onto a highly nonlinear surface. A Nicolas Lancret painting from 1730, *Par un Tendre Chansonnette* (With a Tender Little Song), depicts a man playing a flute for a lady, seated between another woman and a servant boy (Fig. 5). In figure 6, Kelly M. Houle has appropriate rendered Leonardo da Vinci in catoptric anamorphic perspective that one must assume would have been of the greatest interest to the master in his day.

Fig. 5: *A cylindrical catoptric anamorphosis*

Fig. 6: *Leonardo da Vinci, cylindrically restored*

4. Techniques of Anamorphosis

The foundational basis of anamorphosis lies in the concept of perspective as a mathematical transformation. Artists had already developed and refined mathematical techniques in what would later come to be known as projective geometry in order to produce one and two point perspective, but the introduction of catoptric techniques involving conical and cylindrical mirrors opened the door to a need for a whole new set of nonlinear mathematical tools which simply did not exist in those early years, with the invention of the calculus still decades in the future. The artist-geometers of the time simply resorted to modifications of what had worked for linear perspective: “hand” transformation of data from a linear grid field to a nonlinear grid field. Figures 7 and 8 show sketches employed by Jean-François Niceron, a Parisian monk, for creating both oblique (direct) anamorphic translations, and a cylindrical anamorphosis of Saint Francis of Paola, done in 1638. An example of the construction of a conical anamorphosis by Fernandino da Bibena is shown in figure 9, from a book he published in 1731 on perspective and architectural design.

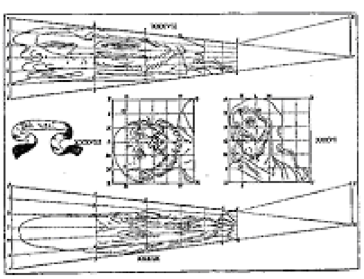

Fig. 7: *Oblique anamorphosis*

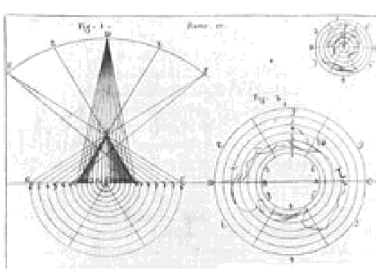

Fig. 8: *Conical anamorphosis*

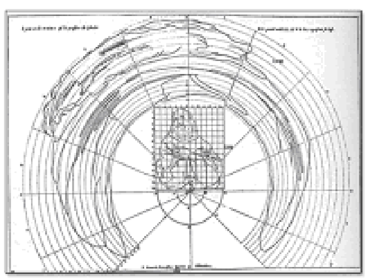

Fig. 9: *Cylindrical anamorphosis*

If one regards the source image as the domain of a mathematical transformation and the "corrected" image (corrected by the transformation) as its range, then perhaps the most common technique for creating conical and cylindrical anamorphoses was to simply paint the domain source image by observing in real time the resultant transformed image in the mirrored cone or cylinder itself. This naturally required considerable hand-eye coordination and a period of adjustment for the brain to learn and comprehend which strokes in the plane would produce desired effects in the reflected image, particularly in the case of a conical transformation where the image was inverted in addition to simply being distorted in lateral dimensions. It should also be noted that a common technique used by artists to create the simpler direct anamorphisms was to draw a normal picture, perforate the outline of the drawing, then place the page at right angles and allow light to shine through the holes onto the new drawing surface. The elongated image could then be traced and painted.

In time, various types of pantograph-like mechanical devices were invented, including machines to assist in drawing the domain images for both cylindrical and conical anamorphoses. A device used to trace an image placed on a cylinder and transcribe it to a distorted source plane image was invented by Jaques Leopold, and a description of it published in Brisson's *Dictionairre raisonné de physique* in 1781 [6]. In modern times, these mechanical machines have been replaced by the computer and software which can produce the domain plane image in the blink of an eye through the power of mathematics. One such example for the PC is a free downloadable program called *Anamorph Me!* [7], and others exist for the Macintosh as well.

5. Modern Examples

Evidence that anamorphic art is alive and well can be found in the work of Colin Wilbourn in Sunderland, England. Wilbourn carved a large scale anamorphic image into a sandstone wall overlooking the River Wear there, completing the work in 1997 as a part of the St. Peter's Riverside Sculpture Project [8]. Although the entire project consisted of three related works relative to the same wall, it is the sculpting of the image of a large door into the wall itself which is of interest in this paper. When viewed from a straight on perspective or from an area to the left of the carving, the image is drastically distorted into a sequence of pinched quadrilaterals, appearing to be the result of giant fingers grasping and pulling on a corner as though the wall had been transformed into a thin latex sheet (Fig. 10). View the wall from the "correct" position, clearly marked by the artist with a chair and keyhole to peek through, and the scene magically transforms into a beautifully carved door with detailed inset work, complete with a stained glass window of a sailing schooner being blown across the waves (Fig. 11). These images and more are accessible at the Public Art Research Archive of Sheffield Hallam University [9].

Figure 10: *'Passing Through' viewed from the 'wrong' side.*

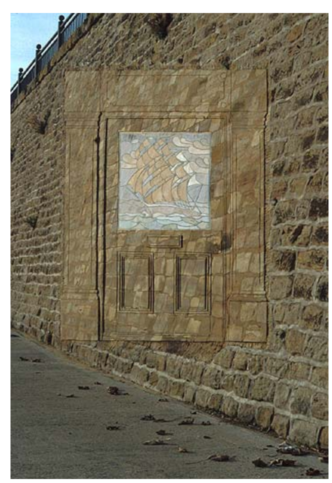

Figure 11: *'viewed correctly'*

Perhaps the most ubiquitous use of anamorphic processing is in the movie industry, where anamorphic lenses are used to compress wide-screen images onto 35 millimeter film, and then again to expand the images for screen projection. Anyone who has sat in a movie theatre in the last five years, or rented a movie on DVD and viewed a wide-screen 16:9 format movie on a standard 4:3 screen has experienced the practicality of anamorphic techniques. Video projection techniques to allow projectors to eliminate keystoning, or to project from oblique angles in a conference room or onto the backdrop of a theatre stage also employ such technology built into the lenses and projection systems. A form of anamorphism even makes an appearance during the daily commute, as the letters "STOP" are usually elongated in the vertical so as to make them intelligible from the low angle of a driver in a car viewing them at a distance.

6. Stereographic Projection

The seed from which this paper grew was the realization that a special transformation in the field of complex variables, first learned by the author as a graduate student decades ago and standard in every textbook on the subject, could actually serve as another type of anamorphic transformation. This is what mathematicians refer to as the stereographic projection, a method for mapping the extended complex plane (consisting of the complex plane $\boldsymbol{C}$, together with the point at infinity, denoted by ∞) onto the unit sphere. Let $\boldsymbol{C}$ represent the complex plane, i.e., the real plane but with the x-axis and y-axis now representing the real and imaginary axes, respectively, so that the complex number $3 + 4i$ is represented in $\boldsymbol{C}$ by the point (3,4). Let $\boldsymbol{H}$ represent a sphere of radius one, with the "south pole" at the origin S(0,0,0), and the north pole at N(0,0,1) in three dimensional space ($\boldsymbol{R}^3$). If we consider a light source located at N, then it can easily be seen that every ray of light emanating from N at an angle below the horizon for N intersects the sphere $\boldsymbol{H}$ in one and only one point P', and the ray from N to P' continues on to intersect the plane $\boldsymbol{C}$ in one and only one point P. If we assign the point N to the point at infinity (∞), then a one-to-one correspondence exists between the extended complex plane and this sphere, referred to as the Riemann sphere in this context (Fig. 12).

Under this transformation, it is straightforward to show that (1) lines of longitude on the sphere project onto rays in the plane emanating from S, (2) lines of latitude on the sphere project onto circles in the plane centered at S, (3) the equator projects onto a circle of radius 2 centered at S, (4) two curves forming an angle θ on the sphere project onto two curves in the plane also meeting at angle θ, (5) lines in the plane transform into circles on the sphere which pass through N, and (6) circles in the plane transform into circles on the sphere (Fig. 13). Thus, the stereographic projection is said to be angle-preserving and circle-preserving. There is very little distortion near S, while any regions near N project far away with enormous distortion, approaching the "infinite edges" of the complex plane as the domain points approach N. In contrast with the more extreme distortion in the "northern hemisphere", when considering the stereographic projection of the "southern" hemisphere, any image features near the equator will be gently spread out with greater detail visible than in comparison with a standard vertical projection onto the plane, which would compress such visual details, making stereographic projections much better suited for two-dimensional display of EEG brain scans.

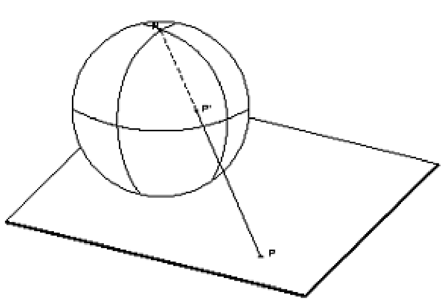

Fig. 12: *The Riemann sphere*

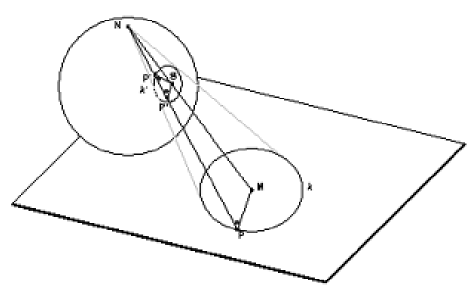

Fig. 13: *Stereographic projection of a circle*

The effect of the inverse of this projection on a logarithmic spiral in the plane is shown by the dotted lines in Figure 14. Although it is not known whether M.C. Escher would have been aware of this transformation (and likely he was not), it is impossible to ignore the remarkable coincidence in form shared by his "Sphere Spirals" from 1958 (Fig. 15).

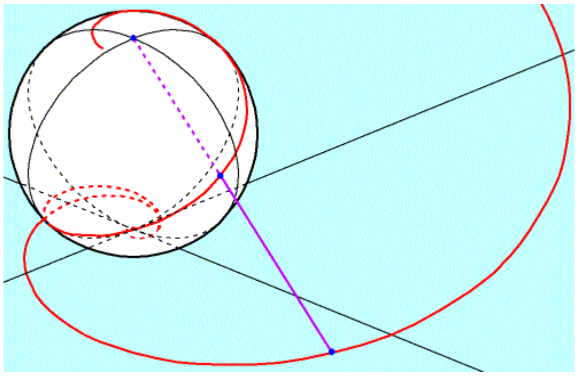

Fig. 14: *Logarithmic spiral on Riemann sphere*

Fig. 15: *Sphere Spirals, M.C. Escher*

7. Visual Interpretation of Infinity

Artists have been concerned for centuries with the expression and embodiment of the infinite in their works, beginning initially with the concept of lines of perspective meeting at the point at infinity, taking on increasingly sophisticated forms in painting and sculpture, and culminating in perhaps the most famous mathematical expression of infinity in Escher's "Angels and Devils", with its underpinnings of hyperbolic geometry, variable metric, and representation of all of two-space within a finite disk. The search goes on for new and more effective ways to transform the unboundedness of the infinite into a more compact model that can be grasped on a human scale. The Riemann sphere and its attendant stereographic projection offer a method for the compression of an infinite two-dimensional space onto a finite spherical shell, which has similarities to H.S.M. Coxeter's original depiction of non-Euclidean geometry, but in a sense performs an inside-outside inversion and expansion into three dimensions.

It is hoped that this mathematical model of projection from one space to another might inspire artists searching for new modes of expression of the infinite, whether they be in the realm of drawing, painting or sculpture, and serve as a seed for further creativity. In the words of the Chandogya Upanishad, "Where there is the infinite, there is joy. There is no joy in the finite".

References

[1] Baltrusaitis, Jurgus. *Anamorphic Art,* p. 33. Harry N. Abrams, Inc. New York, 1977.

[2] Ibid., p. 11.

[3] Vowles, Andrew. News@Guelph. 12 Apr. 1999. Physics Dept., Univ. of Guelph. 29 Jan. 2005. http://www.uoguelph.ca/atguelph/99-04-21/articles/anamorph.html

[4] Baltrusaitis, Jurgus. *Anamorphic Art,* pp. 46-47. Harry N. Abrams, Inc. New York, 1977.

[5] Devices of Wonder (Getty Exhibitions). 16 Nov. 2001. J. Paul Getty Museum. 27 Jan. 2005. http://www.getty.edu/art/exhibitions/devices/flash/

[6] Sharp, John. Count On. 2001. 9 Apr. 2005. http://www.counton.org/explorer/anamorphic/cylindrical-mirror-anamorphosis/

[7] Kent, Phillip. Art of Anamorphosis Software. 23 Apr. 2002. Art of Anamorphosis E-group. 29 Jan. 2005. http://myweb.tiscali.co.uk/artofanamorphosis/software.html

[8] Waterfront Developments: Sunderland. 23 Jul. 2003. Sheffield Hallam University. 26 Jan. 2005. http://public-art.shu.ac.uk/other/sunderland/

[9] Public Art Research Archive. 7 Mar. 2005. Sheffield Hallam University. 15 Apr. 2005. http://public-art.shu.ac.uk/

Figures

1. Ucke, Christian. "Connecting Past and Future". Home page. 10 Feb. 2005. Technical University, Munich, Germany. 28 Jan. 2005. http://www1.physik.tu-muenchen.de/~cucke/ftp/lectures/origprocguilin.PDF

2. Schön, Erhard. Vexierbild, 1535. 27 Jan. 2005. http://physics.uoguelph.ca/morph/schon.html

3. Holbein, Hans. The Ambassadors, 1533. 26 Jan. 2005. http://www.mezzo-mondo.com/arts/mm/holbein/HOH006.html

4. Devices of Wonder (Getty Exhibitions). 16 Nov. 2001. J. Paul Getty Museum. 27 Jan. 2005. http://www.getty.edu/art/exhibitions/devices/flash/

5. Mirror Anamorphs. 3 Apr. 2002. Physics Dept., Univ. of Guelph. 28 Jan. 2005. http://physics.uoguelph.ca/morph/morph1.html

6. Houle, Kelly. Leonardo. 28 Jan. 2005. http://www.kellymhoule.com/about_anamorphosis_frameset.htm

7. Sharp, John. Count On. 2001. 9 Apr. 2005. http://www.counton.org/explorer/anamorphic/plane-anamorphosis/

8. Sharp, John. Count On. 2001. 9 Apr. 2005. http://www.counton.org/explorer/anamorphic/conical-mirror-anamorphosis/

9. Art and Optics: Sidney Perkowitz. 28 Jan. 2005. http://webexhibits.org/hockneyoptics/post/perkowitz.html

10. Wilbourn, Colin. Passing Through, 1997. Sunderland, England. 15 Apr. 2005. http://public-art.shu.ac.uk/other/sunderland/00000031.htm

11. Wilbourn, Colin. Passing Through, 1997. Sunderland, England. 15 Apr. 2005. http://public-art.shu.ac.uk/other/sunderland/fi/00000035.htm

12. Montgomery, Jennifer. Math 309 Term End Project. 2 May 2001. Mathematics Dept., Univ. of British Columbia. 28 Jan. 2005. http://www.math.ubc.ca/~cass/courses/m309-01a/montgomery/

13. Ibid.

14. David, Hollister. Hop's Gallery. 1 Mar. 2005. 16 Apr. 2005. http://clowder.net/hop/Riemann/Riemann.html

15. Escher, M.C. Sphere Spirals, 1958. 27 Jan. 2005. http://www.mcescher.com/Gallery/recogn-bmp/LW428.jpg