(Vector) Fields of Mathematical Poetry

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Abstract

In this note we will look at some artwork inspired by the mathematics of vector fields on surfaces.

1.1. The Art Work. My installation "(Vector) Fields of Mathematical Poetry" will be shown at the Core New Art Space Gallery in Denver from June 2 to June 18, 2005. This exhibition is part of the "2005 CU Special Year in Art and Mathematics" at the University of Colorado/Boulder. This installation was inspired by some beautiful results on vector fields on surfaces that I am going to describe. (All our surfaces are assumed to be orientable.)

1.2. Mathematical Vector Fields on Surfaces. For rigorous definitions, see [1], [3].

a) Surfaces are mathematical objects such as the sphere, and the doughnut's outer shell or a "doughnut with more than one hole"
b) A local vector bundle on a surface is a local assignment of a linear space. Its elements are called vectors and in the easiest cases can be represented by arrows.
c) Local vector bundles patch together to form globally defined vector bundles on surfaces.
d) A vector field on a surface is a smooth assignment of a vector for each point.
e) Vector fields have a well defined index at any of the points at which they are zero. This index denotes the total number of vector field loops around the zero point.

1.3. The Euler Characteristic. The Euler Characteristic X(S) of a planar geometric object S can be computed in the following way. First, subdivide S into triangles (or generalized triangles, triangles which curved edges). Define Then X(S)=T-E+V, where T is the number of triangles, E the number of edges, V the number of vertices in the subdivision. Euler characteristic is additive. The Euler characteristic of the sphere, thought of as a tetrahedron, is 2, while the Euler characteristic of the doughnut is 0.

1.4. Theorems. The following two theorems are classical results

1) The sum of the indices of a vector field on a surface equals the Euler Characteristic.
2) If a surface admits a nowhere zero vector field, then its Euler Characteristic is zero.

The second theorem is a straightforward consequence of the first one, as there is no point at which the index can be computed. Also, as the Euler Characteristic of the sphere is 2, every vector field on the sphere must vanish at a least one point, or, more intuitively, it is not possible to "comb" a sphere without a bold spot. (See Image 1.)

1.5. Proof of Theorem 1. [1], [2], [3]. We will restrict our proof to the sphere and the doughnut. (For general surfaces the proof is similar to these two cases.) Firstly, one can show that Theorem 1 is independent of the vector field chosen. By Morse theory, surfaces are built from the bottom up by performing surgery operations at vanishing points of Morse vector fields. An artistic rendering of Morse vector fields can be obtained by pouring colored viscous liquid from the top of the surface. (See Figures 2 and 3.)

For the sphere, the North and the South Poles are the only zeros of the given Morse vector field. At the South Pole, we attach a half sphere, whose Euler characteristic is 1 (1 is also equal to the index of the Morse vector field at the South Pole). At the North Pole, we also attach a half sphere. The two half spheres are joined together at the equator. Then the Euler characteristic of the sphere, 2, computed by adding the Euler characteristics of the two given halves. And 2 is also equal to the sum of the vector field indices at the North and South Poles.

For the doughnut, the North and the South Poles together with the two points A and B of Figure 3 are the only zeros of the given Morse vector field. At the North and South Poles the situation is similar to the sphere case. At the two points A and B we attach a “pair of pants,” i.e., two cylinders glued together along a point. The Euler Characteristic of a pair of pants is -1, the same as the index of our Morse vector field on it. The Euler Characteristic of the doughnut is therefore 0, computed by adding the Euler characteristic of the two given half spheres to the Euler Characteristic of the two pairs of pants. And 0 is also equal to the sum of the Morse vector field indices at the four zeros.

Illustrations



Figure 1: *It is not possible to “comb” a sphere without a bold spot*
Figure 2: A *Morse vector field on the sphere* **Figure 3**: A *Morse vector field on the doughnut*

References

[1] Victor Guillemin and Alan Pollack, *Differential Topology*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1974.

[2] J. W. Milnor, *Morse Theory*, Annals Studies 51, Princeton University Press, Princeton, 1961.

[3] J. W. Milnor, *Topology from the Differential Viewpoint*, The University Press of Virginia, Charlottesville, 1981.