D-Forms and Developable Surfaces

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Abstract

Every so often you learn of a new concept that is so simple you wonder why it was not thought of before. This paper is about such a case: the exploration of D-Forms, where new three dimensional forms are created by joining the edges of two flat surfaces that have the same length of perimeter. This then leads onto other developable surfaces, all of which offer a host of ideas for sculpture.

Introduction

The concept of D-Forms was invented by the London designer Tony Wills. New surfaces are being discovered as the ideas are explored and this paper just shows a few possibilities. As a product designer Tony has developed such products as the D-Form street furniture range which uses D-Forms as moulds into which is cast artificial stone to create elegant architectural elements and they have been investigated for aircraft propeller shapes [1]. They have much in common with the sculptural forms of artists such as Barbara Hepworth, Constantin Brancusi and Naum Gabo.

D-Forms are three dimensional forms created by joining the edges of two flat surfaces that have the same length of perimeter. The flat surfaces should be made of material that does not stretch or shear. This excludes woven material, this does not mean that the concept cannot be extended in that direction except that the surface will deform. Depending on where you have chosen to start the join the two surfaces, each face 'informs' the other what three dimensional form to finally produce. The emerging D-Form continually changes shape as the edge joining progresses. The final D-Form that results only appears when the process is complete.

Two circles can only join to a flat disk. If one of the surfaces is a circle then only one D-Form results from joining it to another surface. When the perimeter curves of the two surfaces are different, a whole host of D-Forms can be created, depending on which points on the two perimeters are joined. Further possibilities arise when creases are added to the surfaces.

Since D-Forms are a new mathematical discovery, their geometry has not been fully explored. The construction process is remarkable simple, but the mathematical solution of what is happening is remarkably complex. Once they have been constructed as physical objects, it is possible to gain some insight into their properties, but it is not easy to predict the surface properties from the original planar shapes, and so far no one has been able to create them in a computer system other than to approximate them using Evolver [2].

D-Forms are a special type of surface called a "developable surface", which means the surface can be cut open and flattened into a plane. Every developable surface is also a ruled surface, which means that the surface can be defined as the path of a line that moves across the surface. This allows some experimental study of reverse modelling by creating surfaces which can then be developed.

Simple D-Forms

The simplest D-Forms are created from a pair of ellipses. These can be the same ellipse or different ones, although drawing a pair of different ellipses with the same perimeter requires computer software which can match the perimeters. There is no formula for calculating the perimeter of an ellipse.

It is not easy to appreciate the way a D-Form grows before your eyes as you construct it unless you carry out the process. There is as much beauty in making a D-Form as there is in the sculptural sensuality of the finished form. To comprehend this, I suggest you try making some with the same ellipse for both surfaces being joined. Even if you do not have computer graphics software, it is easy to create an ellipse by drawing a circle in the graphics feature of a word processor and then squashing it in one direction to make the ellipse. Photocopier paper is ideal for making them.

The fastest way to make D-Forms is to join the edges together by using small pieces of sticky tape. You need to keep the width small so that the edges of the two shapes join easily and are not distorted. The use of a heavy but small tape dispenser (12mm 'invisible' drafting tape is ideal) is highly recommended. I have also made models where the edges are joined with overlapping "fingers" but this forces the joining in a discrete number of ways. Tony Wills has also worked with thin metal and used a similar technique, using laser cutting. The surfaces can be quite large. Figure 1 shows Tony Wills with some steel D-Forms and you can see the joining technique along the edges.

Figure 1: *Tony Wills with steel D-Forms*

One of the most interesting aspects of D-Forms is the range of surfaces obtainable from a single pair of shapes. A good illustration of this is shown in figures 2 and 3. The D-Forms are all created from the same pair of ellipses. The left D-Form of figure 2 has the end of the minor axis of one ellipse joined to the end of the major axis of the other, so forming the D shape. At the right, the two ellipses are rotated slightly before joining. This gives a slight twist which increases (figure 3) until the two ellipses are planar.

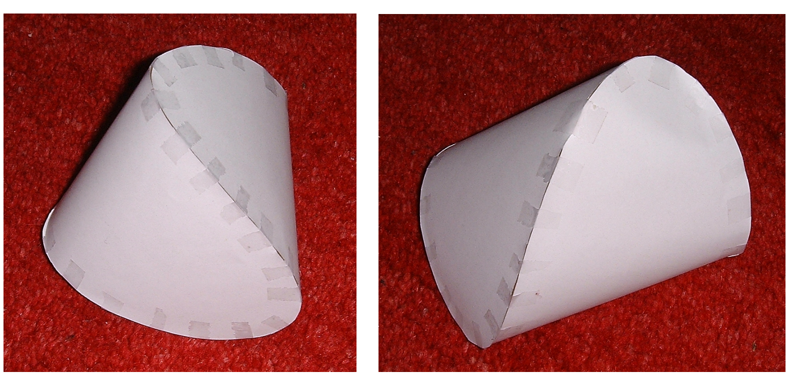

Figure 2: *D shape plus slight twist*

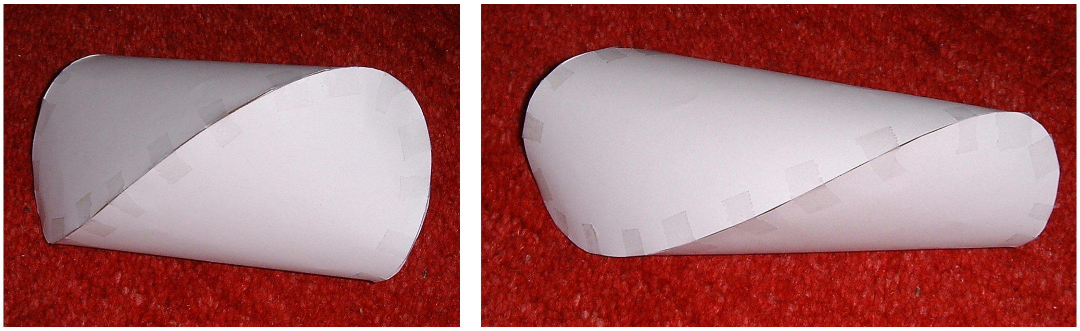

Figure 3: *further rotations with more pronounced twists*

There are obviously many more possibilities using other curved shapes as well as multiple ones, but there is no space to show any more, since I need to consider other variations.

D-Forms with creases

Some shapes seem ideal to attempt to make a D-Form from, but they don't always behave as you might expect. For example, if you take a circle and a square with the same perimeter, the circle does not have a preferred state and crumples. However, if you crease the circle along two perpendicular diameters, the resulting D-Form takes on a definite shape. When Tony Wills discovered it, he called it the "Squaricle". Figure 4 shows the square and circle in the correct proportions together with a brass version made for Tony Wills.

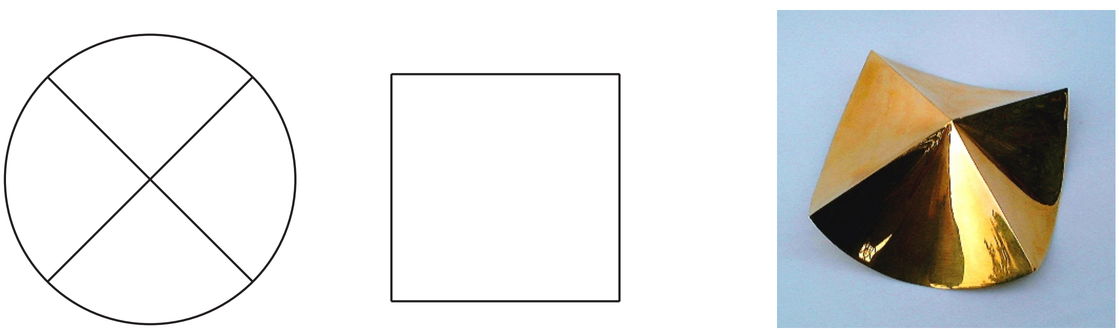

Figure 4: *Squaricle*

There are many more possibilities using this creasing concept. For example, rectangles and other polygons could be used. The Squaricle is one D-Form surface that can be analysed [3]; and it is found to be a set of four sections of a cone from the circle and a planar square (with vertices which are the mid-points of the square) together with four parts of cylinders from the corners of the square.

Anti-D-Forms

As originally conceived, D-Forms are created by joining the outside edge of two shapes with the same perimeter. In writing a book on D-Forms [3], I also considered what I called anti-D-Forms where holes are cut in surfaces and these inner edges are joined in the same way. Figure 5 shows various views of the same anti-D-Form.

Figure 5: *Anti-D-Forms from elliptical holes*

There is also an intermediate case where the outside edge of a shape is joined to the inner edge of a hole and further refinements such as adding creases can also increase the possibilities. Figure 6 shows some variations by Tony Wills using the squaricle concept.

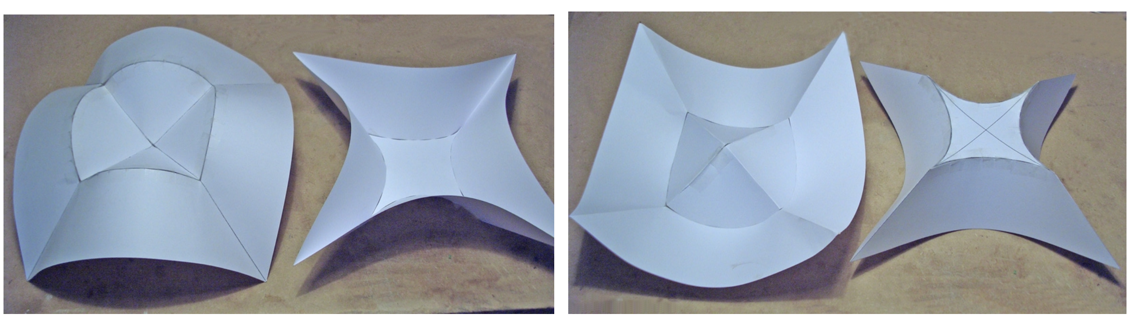

Figure 6: *Anti-D-Form squaricles*

We have also explored the possibilities of using the outside edges as part a further D-Form construction.

Developable surfaces and ruled surfaces

As I said above, D-Forms are a special type of surface called a “developable surface”. Strictly speaking developing a surface means bending it so as to change its form, but the term “developable surface” is commonly restricted to such inextensible surfaces that can be developed into a plane, or, in common language, “ smoothed flat”. This process is in the direction of taking the surface (which may need to be cut open) and flattening it into a plane. D-Forms are perhaps better described as “applicable surfaces” since the process is one of taking a pair of plane surfaces and bending them to a pair of curved surfaces with a common edge.

Another property of developable surfaces is that they are also ruled surfaces. It should be stressed that whereas all developable surfaces are ruled surfaces, the converse is not true. For example, the hyperboloid and hyperbolic paraboloid which are probably the best known ruled surfaces are *not* developable.

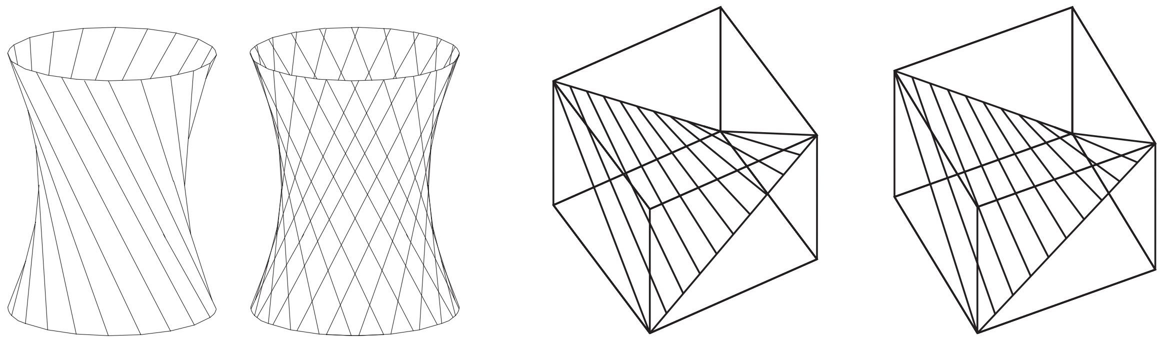

Figure 7: *hyperboloid and hyperbolic paraboloid (as a stereo pair)*

The condition for a ruled surface to be developable is that two adjacent rulings must intersect (at infinity in the case of a cylinder) so adjacent rulings must lie in the same plane. The hyperboloid and hyperbolic paraboloid are both double ruled surfaces. In the case of the hyperboloid, a single and double ruling is shown. In either case, only one set of rulings (generators) need to be considered for development. It is adjacent rulings of each set which must intersect, not rulings of different sets. If you imagine the development being one of rotation about the ruled line acting as a hinge then it is easy to see that the lines must intersect in order that such a hinging will result in a flat plane. The hyperboloid and hyperbolic paraboloid are both generated by rulings which are skew, and so they are not developable. This is more evident in the case of the hyperbolic paraboloid.

The consequence of this is that most D-Forms consist of pairs of surfaces which are either parts of cylinders or cones. Cylinders are special cases of cones where the vertex of the cone is at infinity. (*Note that the cylinders are not necessarily circular cylinders nor are the cones necessarily right circular cones*.) The rulings in the surface of a cone intersect at the vertex (figure 8).

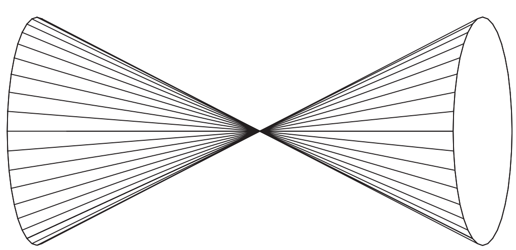

Figure 8: *cone with ruling lines*

This shows why the cone is the most well known developable surface. Since the surfaces of most D-Forms are either parts of cylinders or cones, then if you take a ruler and move it over the surface, the ruler either moves in a direction perpendicular to the rulings or rotates about a point. Figure 9 is a triple surface D-Form which is clearly seen to consist of two conic surfaces and a cylindrical surface.

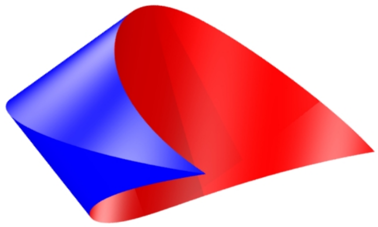

Figure 9: *a crescent D-Form*

Whereas it is not possible to compute D-Forms by starting with two planar shapes, because they are either intersections of cylinders or cones, it is possible to “reverse-engineer” some shapes by for example taking two cylinders and intersecting them and then developing the result. An example of such a D-Form is shown in figure 13.

The Möbius strip paradox

Taking a strip of paper and twisting the ends and joining them creates what is probably the most famous developable surface, the Möbius strip or band. This is often described mathematically as a combination of two rotations of a segment of a line. Take a line segment which starts off parallel to an axis and rotate the segment about the axis while at the same time rotating the segment about its centre as it goes round the circle of revolution formed by the centre.

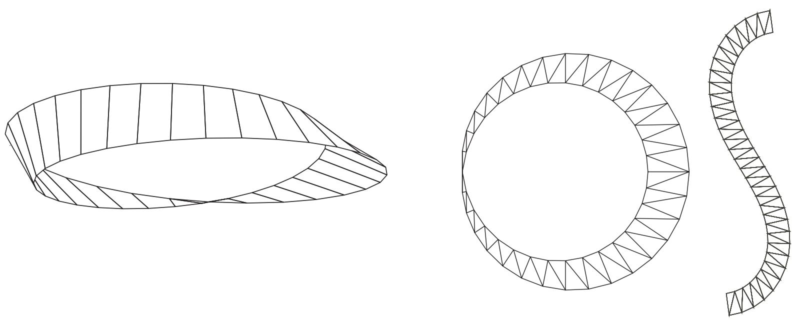

Figure 10: *Möbius strip*

So we have a paradox that as everyone knows the Möbius strip to be created as a developable surface but the common textbook illustration as shown at the left of figure 10 is not developable even though the surface is obviously a ruled surface! Adjacent lines of the surface generated in this way do not intersect but are skew by the very nature of the rotation about their centre. However, if you triangulate the surface, then it is possible to develop it using software like Pepakura [4] as shown in the right of figure 10. This compounds the paradox, and there are many cases in books of sheet metal work which show theoretically non-developable surfaces which are developed for example Abbott, p 279.

Tangential developables

If the generating lines of a developable ruled surface are not parallel they must meet since they are in a plane when the surface is plane. If they do not meet all in one point, they must meet in several points and in general, each one meets its predecessor and its successor in different points. Such a surface is called a tangential developable since the developable lines will in general be tangents to a curve (the locus of the points of intersection when the number is infinite). This curve is called the edge of regression of the surface. In the case of a cone, the edge of regression is a point which is at infinity for a cylinder.

One of the ways to create a model of tangential developable is described by Thompson and Tait [5] as follows with their figure redrawn as figure 11.

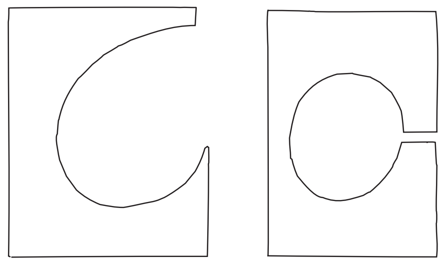

Figure 11

“To construct a complete developable surface in two sheets from its edge of regression. lay one piece of perfectly flat, unwrinkled, smooth-cut paper on the top of another. Trace any curve on the upper, and let it have no point of inflection, but everywhere finite curvature. Cut the two papers along the curve and remove the convex portions. If the curve traced is closed, it must be cut open (see second diagram). Attach the two sheets together by very slight paper or muslin clamps gummed to them along the common curved edge. These must be SO slight as not to interfere sensibly with the flexure of the two sheets. Take hold of one corner of one sheet and lift the whole. The two will open out into the two sheets of a developable surface, of which the curve, bending into a curve of double curvature, is the edge of regression. The tangent to the curve drawn in one direction from the point of contact, will always lie in one of the sheets, and its continuation on the other side in the other sheet. Of course a double-sheeted developable polyhedron can be constructed by this process, by starting from a polygon instead of a curve.”

If only they had not cut the curve open and rotated it slightly, they would have discovered anti-D-Forms, which would probably have taken them on the path to D-Forms some 120 or so years ago. Their method is also described in Koenderik [5] who says to take a series of circles on two sheets of paper. These form the tangential developable helicoids of a helix which is the edge of regression. He shows how the two halves of the tangents to the circles form the two surfaces which meet at the edge of regression (figure 12).

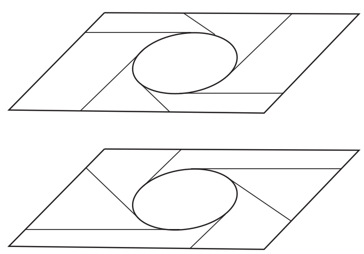

Figure 12

The helicoids are not as prominent as in the anti-D-Forms of figure 5. Tangential developables are easy to program as ruled surfaces. If you have a space curve, you can just extend the tangents. Figure 13 shows a D-Form which has been created by the intersection of two elliptical cylinders. The edge curve was then extracted from this and the two tangential surfaces created as shown as two views of figure 14.

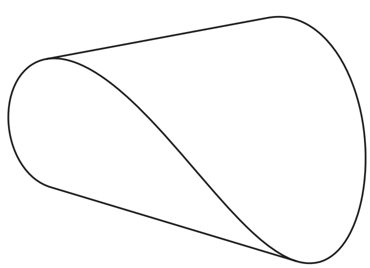

Figure 13: *simulated D-Form*

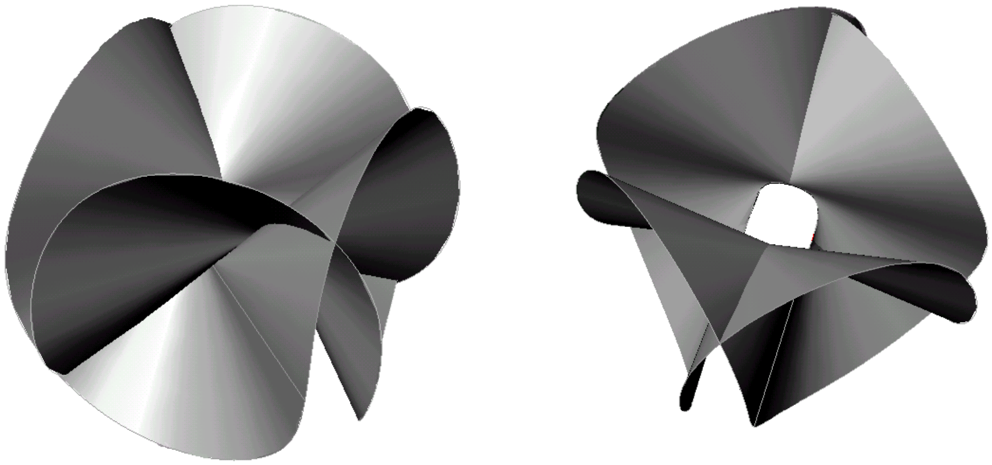

Figure 14: *tangential developable*

The two surfaces result from the two halves of the tangent of the space curve. Thompson and Tait and Koenderik's example of figure 12 took two separate circles and produced the edge of regression by twisting to form the space curve. When you take a space curve as in the edge curve extracted from the example of figure 13, the two tangent halves create the surfaces. This is not an intuitive result, and like constructing D-Forms, needs to be performed to be understood. What is interesting in this case (and also counter-intuitive) is that the two tangential developable surfaces intersect one another.

Concluding thoughts

The more you explore this simple concept the more you wonder why it was not thought of before. There has only been space to touch on the fundamentals of the subject in this paper. It has also not been possible to show the dynamic way a D-Form takes on a life of its own as is constructed. The only way you can understand this is to make one for yourself. The subject raises many questions and suggests many avenues to explore.

References

1. Tony Wills has examples of his D-Form work at
   www.wills-watson.co.uk/proj_street_01.html
   Figure 1 is copyright Tony Wills 2004 as is the Squaricle shown in figure 4.
2. Paul Bourke http://astronomy.swin.edu.au/~pbourke/surfaces/dform/. He may reorganise his site, so this may change.
3. John Sharp, "D-Forms: surprising new 3D forms from flat curved shapes", Tarquin 2005
4. Jun Mitani, Pepakura Designer, www.tamasoft.co.jp/pepakura-en
5. Sir William Thomson (Baron Kelvin) and Peter Guthrie Tait, "A Treatise on natural philosophy", 1888
6. Jan J. Koenderik, Solid Shape. MIT 1990
7. W Abbott, "Practical Geometry and Engineering Graphics" Blackie London1943