BRIDGES
Mathematical Connections
in Art, Music, and Science

From Hot Spots to High School Geometry and Calculus

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Abstract

We discuss the Hot Spots Conjecture, and some connections to elementary ideas in geometry and advanced calculus.

1. Introduction

Eigenvalue problems occur throughout applied mathematics, physics, engineering and finance, as well as music and the art. The important need of their study is recognized and appreciated in all the scientific disciplines where they frequently occur.

Eigenvalue problems might be mathematically challenging, nevertheless they are suitable for presentation in classroom instruction, as their applied significance is easily understandable. In addition, their frequent connection to relatively simple geometry and advanced calculus problems, makes them appealing for a presentation connecting real word problems, physics and mathematics.

We illustrate the above point, by looking at a well-known eigenvalue problem still not-completely resolved: the so-called Hot Spots Conjecture. The problem can be introduce even to a non-experts audience via its clear simple physical meaning: Do the hottest and coolest spots in a perfectly insulated region necessarily move toward the boundary? We proceed by briefly explaining the applied significance of the answer to the question, review the known results, and write down the mathematical formulation of the problem, the later accessible to anyone with an advanced calculus background. We then consider numerical simulations of the problem, using the PDE-Toolbox of the commercial software Matlab. The goal is by looking at the geometry of the numerics to formulate geometric results, that can be proven by means of elementary Euclidean geometry. We also show how basic calculus tools and sophisticated but intuitive mathematical ideas, can be applied to derive a new understanding of the behavior of the temperature profile for large times, for the case of particular type of domains.

2. Background

Some basic geometric understanding of eigenvalues was already known to the ancient Greeks who discovered the inverse relationship between the length of a string and its pitch. A string which is half as long as another is said to produce the same note, only one octave higher and vibrating twice as fast. The physical study of vibrating strings was carried out predominately by Galileo, Bernoulli, Euler and many others more than two centuries ago. The study of two-dimensional eigenvalue problems is much more complex, and continues to the present day for even simple sounding problems. In 1787 Ernst Chladni experimented with patterns formed by vibrations of plates, impressing Napoleon enough to offer a prize for a mathematical theory explaining the Chladni's experiments, [8]. This prize was ultimately claimed by a French mathematician, Sophie Germain in 1816. Two-dimensional eigenvalue problems are at the heart of problems in wave propagation, heat conduction and electromagnetism. H. Weyl obtained various formulas relating the set of eigenvalues or fundamental frequencies to geometric quantities such as area, leading Mark Kac to ask in 1966 whether the geometry of a domain could be obtained based on its eigenvalues. The popularization of this idea is whether one can hear the shape of a drum. This question was answered in 1994 by Carolyn Gordon and coauthors with the discovery of two domains which could be cut and pasted onto one another so as to preserve the tones but not the geometry, [2]. Interestingly, it is still unknown if convex domains are determined by their sound.

A very recent problem of two-dimensional geometry is the so-called Hot-Spots conjecture, [7]. The problem asks whether the hottest and coolest spot in a perfectly insulated region necessarily tend toward the boundary. If one discusses arbitrary regions, it is known that the Hot Spots conjecture is false [1], but there is intense continuing work for the case of regions without holes. The question can be translated in terms of maximum and minimum values of eigenfunctions of the associated Neumann eigenvalue problem for the Laplace operator.

We look at the geometry of level sets for some eigenfunctions of the Laplacian in a region which is almost perfectly insulating. This case known as Robin eigenvalue problems serves as an intermediary between the Neumann-perfectly insulated and the Dirichlet-perfectly conducting. We study the problem numerically, and give an example of how the geometry of the numerics and deep mathematical results lead to a simple geometric conjecture, which can be easily shown in Euclidean geometry.

We also present a more advanced new analytical result which gives an insight on the long term behavior of the temperature profile, where use of a calculus knowledge and intuitive ideas should result in an appreciation, from the part of the audience, of the power of mathematical tools in understanding nature's behavior.

3. Hot Spot Conjecture

The temperature flow, $u(x,t)$, in an insulated, bounded, two-dimensional medium, denoted by Ω, in suitable rescaled coordinates, verifies the following initial-boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \Delta u, & \text{in } \Omega, \quad t > 0 \\ \frac{\partial u}{\partial n}(x,t) = 0, & \text{on } \partial\Omega, \quad t > 0 \\ u(x,0) = f(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where $x = (x_1, x_2)$, $\Delta u \equiv \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$ is the Laplace operator, u verifies the Neumann boundary condition for any $t > 0$ (alias it has zero normal derivative on the boundary), and $f(x)$ is the initial temperature profile.

The long term asymptotic behavior (the so-called steady-state) of u tends to a constant, namely

$$u(x,\infty) = \frac{1}{|\Omega|}\int_\Omega f(x)\,dx = C_{ave},$$

which can be related to the first eigenvalue of the Neumann eigenvalue problem for the Laplace operator, i.e. the smallest μ for which (2) below has a non-zero solution:

$$\begin{cases} -\Delta\phi(x) = \mu\,\phi(x), & \text{in } \Omega \\ \frac{\partial\phi}{\partial n}(x) = 0, & \text{on } \partial\Omega. \end{cases} \tag{2}$$

In fact, one can show that the first eigenvalue is $\mu_1 = 0$, and that the corresponding eigenfunction is a constant, say $\phi_1 = C_{ave}$.

The next leading contribution to u for t large comes from a suitable linear combination of linearly independent eigenfunctions, corresponding to the second eigenvalue μ_2 i.e. solutions of (2), with the smallest $\mu \equiv \mu_2 > \mu_1 = 0$, that is the solutions of the freely vibrating membrane problem. As a consequence, for large t the extreme values of u are roughly achieved at the same locations of the extreme values of the mentioned linear combination.

We run a numerical simulation of (1), for the initial heat distribution shown in Figure 1. Figure 2. shows the level sets of u for large times; since the domain chosen has only one linearly independent eigenfunction, ϕ_2, the level sets in Figure 2. roughly coincide with the ones of ϕ_2.

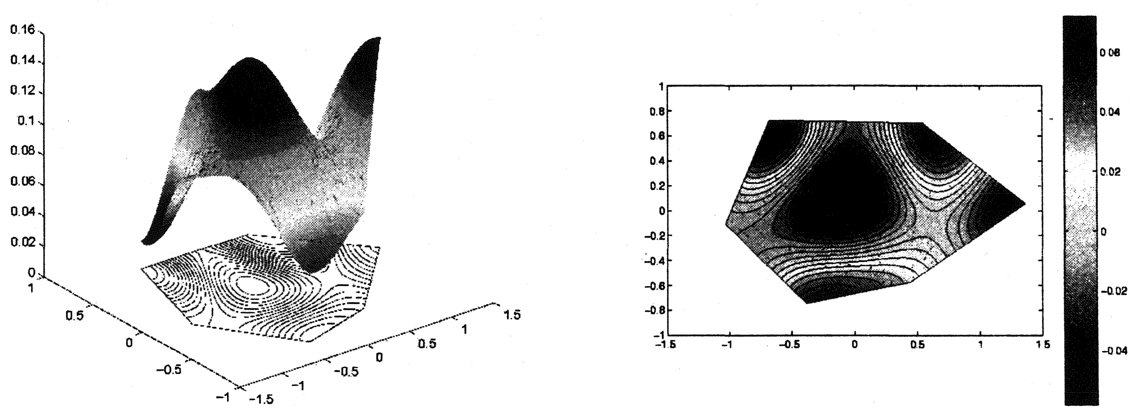

Figure 1: *An initial heat distribution and its level sets*

The statement of the Hot Spot Conjecture, which tries to answer the question of where the hottest and coolest spot for large times are located, in terms of the second eigenfunctions of (2) takes the following form:

Hot Spots Conjecture: *If ϕ_2 is an eigenfunction for (2) corresponding to μ_2, then*

$$\max_{x\in\overline{\Omega}} |\phi_2(x)| = \max_{x\in\partial\Omega} |\phi_2(x)|$$

i.e. the maximum and minimum values of ϕ_2 occur on the boundary.

The conjecture was shown to be false in general for arbitrary regions by Burdzy and Werner in [1]. On the other hand, a major interest in physics surrounds the still open question on whether the conjecture is true for domains without holes.

From an application point of view, it is also of interest to know how fast the solution approaches the constant temperature steady-state profile. The rate is given by an exponential decay of the

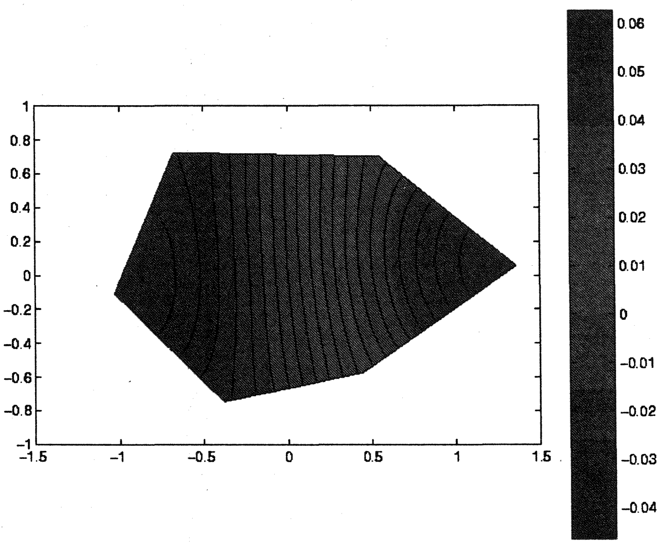

Figure 2: *Level sets of u for $t >> 0$, for the initial heat distribution in Figure 1.*

form $e^{-\mu_2 t}$. Estimates of μ_2 are thus of interest, and much geometric work has been done in this direction by e.g. Payne [4], Sperb [6] etc.

4. Robin Eigenfunctions and High School Geometry

In dealing geometrically with the Hot Spots conjecture, one of the difficulties arises from the absence of levels sets for the first eigenfunction, which we recall is a constant: $\phi_1(x) = C_{ave}$. This is the case, because, as the first eigenfunction is orthogonal to the second eigenfunctions, by looking to the level sets of it one gains an insight on the level sets of the second eigenfunctions.

To overcome this difficulty, we consider the so-called Robin problem, which for α small can be interpreted as a perturbation of the Neumann problem (1):

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \Delta u, & \text{in } \Omega, \quad t > 0 \\ \frac{\partial u}{\partial n}(x,t) + \alpha\, u(x,t) = 0, & \text{on } \partial\Omega, \quad t > 0 \\ u(x,0) = f(x) & \text{in } \Omega \end{cases} \tag{3}$$

To consider this perturbation is also reasonable from the physical point of view as perfect insulators in practice do not exist.

In contrast to the situation of a perfect insulator, the solution of the Robin problem (3) slowly tends to zero, as heat escapes through the boundary via the αu term, and tends to look like $e^{-\lambda_1(\alpha)\, t}\, \psi_1(x)$. Here, $\psi_1 > 0$ is the first eigenfunction, of the Robin eigenvalue problem again for the Laplacian:

$$\begin{cases} -\Delta\psi = \lambda(\alpha)\, \psi, & \text{in } \Omega \\ \frac{\partial \psi}{\partial n}(x) + \alpha\, \psi(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{4}$$

We denote by $\lambda_1(\alpha) > 0$ the smallest eigenvalue. Note that $\lambda_1(\alpha)$ is no longer zero as with the Neumann case, and ψ_1 is not constant, therefore, the hottest and coolest spots of u solution of (3) will tend to the ones of ψ_1.

We want to point out that as heat escapes from the boundary, in the Robin boundary condition case the coolest points are on the boundary, but the hottest are not. They will be in general in the inside of the domain. So, why is it of interest to study the Robin Boundary condition in this context? The reason lies on the fact that for a fixed domain Ω, the second eigenfunctions of the Robin problem (4) are very similar to the second eigenfunctions of the Neumann problem (2); hence, by looking at the level sets of ψ_1, we gain an idea of the level sets of the second eigenfunctions of (2).

The hottest spot for a triangular region, computed by numerically solving (4) for $\lambda(\alpha) = \lambda_1(\alpha)$, and plotting the levels sets of ψ_1, can be seen as the center of the circles in Figure 3.

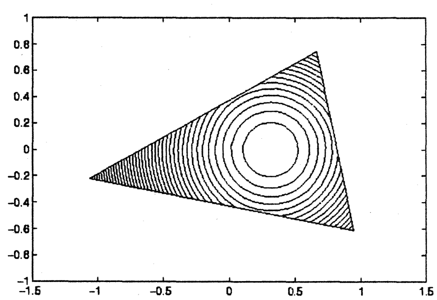

Figure 3: *The hottest spot is the center of the shown circles*

We considered similar numerical simulations for various polygonal regions both convex or non-convex, some results are shown in Figure 4.

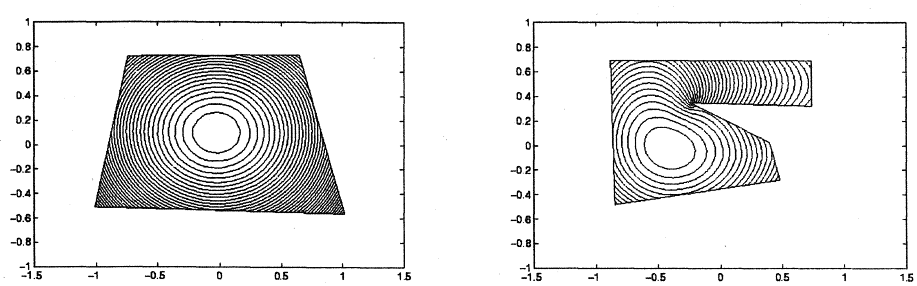

Figure 4: *The hottest spot is the center of the oval regions*

It turns out (see Sperb [6]) that while $\lambda_1(\alpha) \to 0$ as $\alpha \to 0$ (which is expected as the first eigenvalue of the Neumann problem is zero), the ratio of the first eigenvalue to the parameter α tends to a geometric quantity

$$\lim_{\alpha \to 0} \frac{\lambda_1(\alpha)}{\alpha} = \frac{L}{A},$$

where L is the arclength of Ω and A its area.

By looking at the level sets of the long time behavior for various different domains, we notice that the level sets are circular, precisely when the region Ω is a polygon circumscribing a circle. Since the inscribed circle is a level set for ψ_1 which touches the boundary, the first eigenvalue of (4) for this circle, should be very close to ψ_1.

From this remark and the quoted result of Sperb, one is lead to argue that the limiting ratios $\frac{\lambda_1(\alpha)}{\alpha}$ are the same for the polygon, and the circle it circumscribes. But for a circle of radius $1/K$ it holds $\lim_{\alpha\to 0}\frac{\lambda_1(\alpha)}{\alpha} = 2K$, and these observations can be summarized in a simple geometric conjecture which can be verified using high-school level geometry.

A Geometric Result: *If P is a polygon which circumscribes a circle C with radius $1/K$, then the perimeter L of P divided by the area A of P is twice the curvature, that is $L/A = 2K$ or $L = 2KA$.*

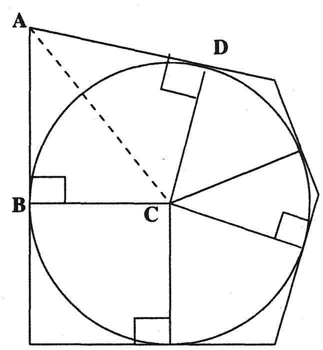

Figure 5: *Inscribed circle*

A straightforward proof of the above can be given as follows. One first divides the polygon into quadrilaterals such as ABCD in Figure 5., from which the student can see that it is enough to show that length(AB)+length(AD) = 2 K Area (ABCD). Since length(AB)=length(AD), one needs only to derive length(AB)= K Area(ABCD). This can be done, if one observes that Area(ABCD)=Area(ΔABC) + Area(ΔACD)= 2 Area(ΔABC) =2 $\frac{1}{2}$ length(AB) length(BC), as $\angle$ABC is a right angle. But, length(BC)=$1/K$, and we have Area(ABCD)=2 1/2 length(AB) $1/K$; which is the desired result.

5. A More Intricate Study of the Decay for Ω Convex

The Robin problem (3) is characterized by heat escaping, and the first order term tends to behave approximately as $e^{-\lambda_1(\alpha)\,t}\,\psi_1(x)$. A measure of how good is this approximation is given by the so-called gap between the first and the second eigenvalue, $\lambda_2(\alpha) - \lambda_1(\alpha)$.

For convex regions, with first eigenfunction having convex level sets one can find a lower bound on how fast the heat distribution tends to $e^{-\lambda_1(\alpha)\,t}\,\psi_1(x)$, by a mix of basic calculus tools and instructive intuitive ideas.

A Calculus Result: *If Ω is a convex region, whose first Robin eigenfunction $\psi_1(x)$ has convex level sets then $\lambda_2(\alpha) - \lambda_1(\alpha) \geq \frac{\pi^2}{d^2}$, where d is the diameter of Ω.*

We follow an argument presented in [5]. Using calculus, the eigenvalue gap $\lambda_2(\alpha) - \lambda_1(\alpha)$ can

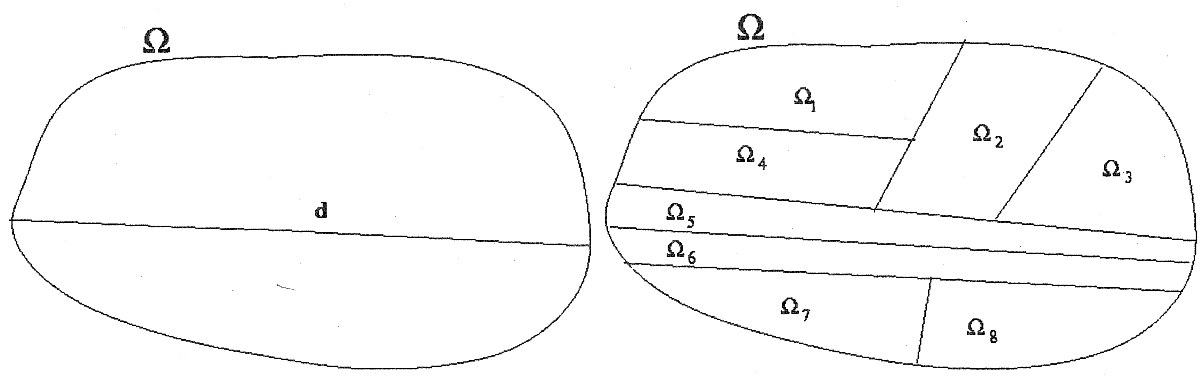

Figure 6: *Region Ω - Subdivision of Ω*

be represented as a minimum (of an energy):

$$\lambda_2(\alpha) - \lambda_1(\alpha) = \min \frac{\int_\Omega |\nabla f|^2 \psi_1^2}{\int_\Omega f^2 \psi_1^2},$$

where f is any differential function on Ω with $\displaystyle\int_\Omega f\,\psi_1^2 = 0$, and the minimum is achieved for $f = \dfrac{\psi_2}{\psi_1}$.

One can easily visualize the fact that for every angle $0 \le \theta \le 2\pi$ or equivalently for every direction, there is a unique line l_θ parallel to that direction which divides Ω into regions of equal area, one to the left of l_θ, one to the right, say L_θ and R_θ, respectively. Since $l_\theta = l_{\theta+\pi}$, if $\displaystyle\int_{L_\theta} f\,\psi_1^2 > 0$, then $\displaystyle\int_{L_{\theta+\pi}} f\,\psi_1^2 = \int_{R_\theta} f\,\psi_1^2 < 0$.

By continuity, there must be $\theta, l_\theta, L_\theta, R_\theta$, with $\displaystyle\int_{L_\theta} f\,\psi_1^2 = 0$. Repeating the process, we can further divide the regions (see Figure 6., where Ω is divided in 8 pieces of equal area, each piece being convex and with $\displaystyle\int_{\Omega_i} f\,\psi_1^2 = 0$.)

Invoking the Mean Value Theorem, we have that in the limit our result is proven if we show that $\int_l |f'|^2 \psi_1^2 \ge \frac{\pi^2}{d^2} \int_l f^2 \psi_1^2$, where l is an arbitrary line segment in Ω, and $\displaystyle\int_l f\,\psi_1^2 = 0$. Again using calculus, one can characterize the above as an eigenvalue problem, by showing that

$$\mu_{\psi_1} = \min_{\int_l f\,\psi_1^2 = 0} \frac{\int_l |f'|^2 \psi_1^2}{\int_l f^2 \psi_1^2},$$

is the smallest eigenvalue of the one-dimensional Neumann eigenvalue problem

$$\begin{cases} [\psi_1^2 v]' + \mu \psi_1^2 v = 0, & \text{in } l \\ v' = 0, & \text{at the endpoints of } l. \end{cases} \tag{5}$$

The standard change of variable $w = v'\,\psi_1$, lead to the new problem

$$w'' + \left[\frac{1}{2} \frac{(\psi_1^2)''}{\psi_1^2} - \frac{3}{4} \frac{[(\psi_1^2)']^2}{(\psi_1^2)^2} \right] w + \mu w = 0,$$

with $w = 0$ at the endpoints of l.

Then integration by parts, using the fact the ψ_1 is a concave function on l, gives the result.

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