

An Introduction to the Tiling Properties of the Tangram and Their Application in Two and Three Dimensions

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Abstract

This introduction to the tiling properties of the Tangram focuses upon the property called Preciousness. It continues with examples of the unique way in which they can produce an infinite number of tiling patterns. It explains the iterative nature of the process as applied to designs in two and three dimensions. Summaries of the Geometric properties and Artistic examples are included.

1. Introduction

This work follows on from previous work on Precious Triangles [4], [5]. The idea of Preciousness has been expanded to include polygons. This paper concentrates upon the tiles forming the Tangram. Most people will be familiar with the standard Tangram puzzle. The origins of the Tangram are uncertain, but it was popular in China. There is evidence of similar tiles being around in Babylon around 2000 B.C. [1]. Archimedes is credited with a similar puzzle around 250 B.C. [2]. Tangram tiles can be cut from a square as indicated in figure 1. The usual rules are that all seven tiles should be used and they should touch but not overlap another tile. Some examples can be seen in figure 1. Many others can be found on the internet. [3].

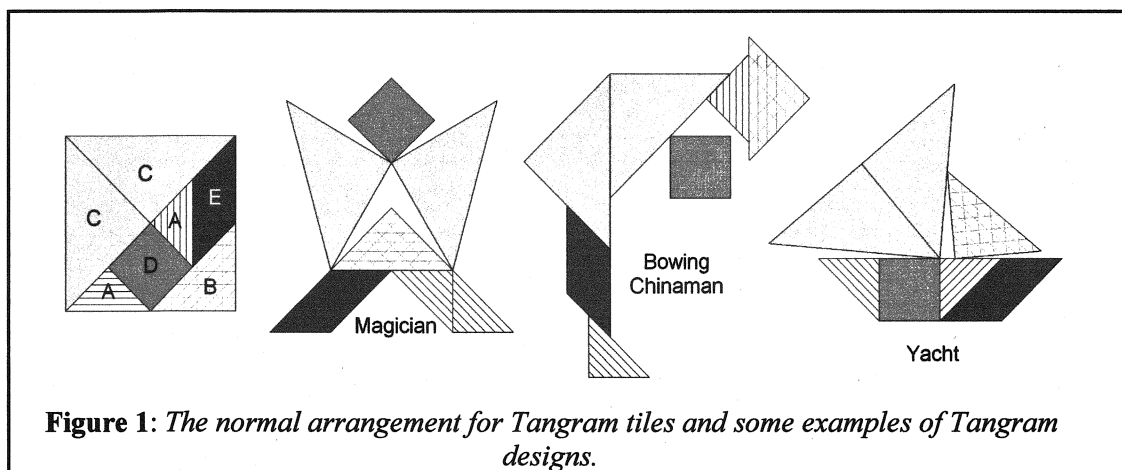


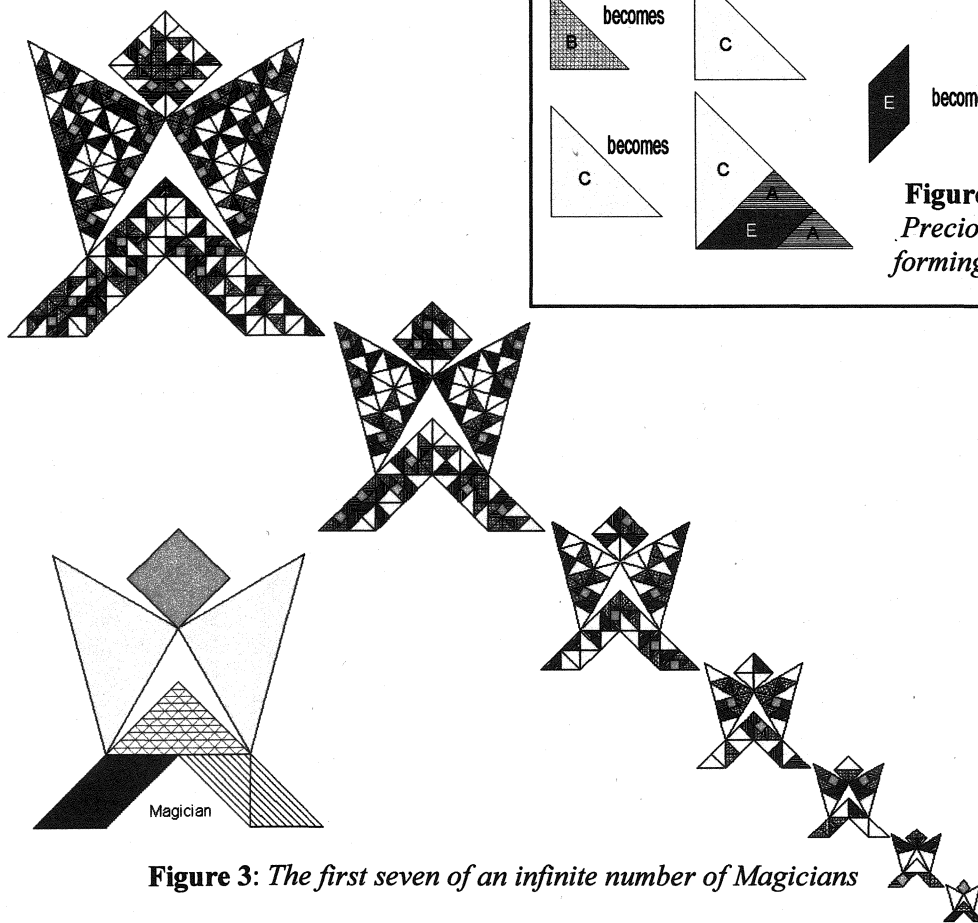
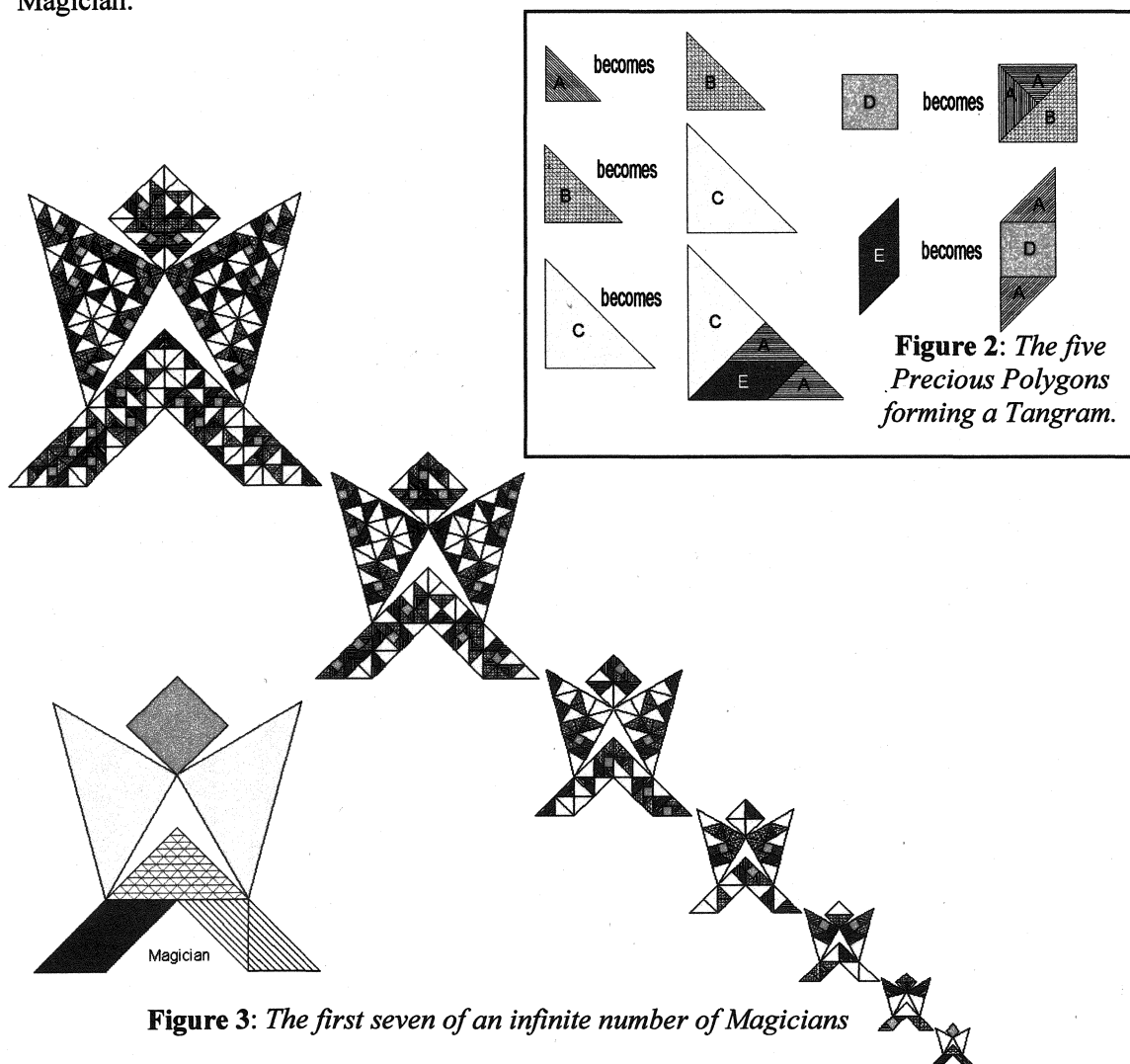
Figure 1: *The normal arrangement for Tangram tiles and some examples of Tangram designs.*

2. What are Precious Polygons.

Precious Polygons are sets of different polygons which can be used to form other sets of similar polygons. A process similar to that of Solomon Golomb's Rep-tiles [7] but involving sets of polygons rather than a single polygon. We shall see later that this will allow us to create an infinite series of designs. A number of such sets with the property have been identified. One of these is the set that forms the Tangram square shown in figure 1.

3. Precious Polygons from the Tangram.

There are five different polygons that form the tangram square in figure 1: three triangles, one square and a parallelogram. Figure 2 shows a scheme whereby each of the original polygons can combine together to form a similar but larger version of the original set. It can be quite easily shown that each large version is $\sqrt{2}$ larger than the original [4],[5]. This enlargement factor must be a constant for the set to be Precious and is known as the Precious Ratio. Now that we have a Precious set then we can take any design made from the original Tangram shapes and produce a larger version using the scheme in figure 2. Since we end up with a design using only the original shapes we can repeat the process ad infinitum. Each successive design is larger than the previous one by a factor of $\sqrt{2}$. Figure 3 shows the development of the series of designs based upon the Magician.



4 Some Interesting Ratios

An interesting aspect of Precious properties of the Tangram tiles is that, no matter how many of each tile in the original design, after a few generations you always end up with the same proportion of each type. It can be shown that the proportion of each approaches $7/18$ for tile A, $2/9$ for B, $2/9$ for C, $1/18$ for the square and $1/9$ for the parallelogram. A proof for this can be found at ref[4]. The more generations there are, the closer this becomes.

5 Artwork in Two Dimensions

The simplest type of two dimensional artwork is the standard Tangram design and the generation of a series each being $\sqrt{2}$ times larger than the one before (see figure 3). The second, called selective transformation, is a similar process but not all the polygons are transformed. The third is the creation of a mosaic using the Tangram shapes as tiles.

6 Selective Transformation

In this example, square D, once it is created, is not transformed (see figure 2). In figure 4 the polygons of the fish design are transformed as normal, except for the square which contains a picture of sterling currency instead of being transformed. Details of generation 0 and generation 2 are shown in the bottom left and top right of figure 4.

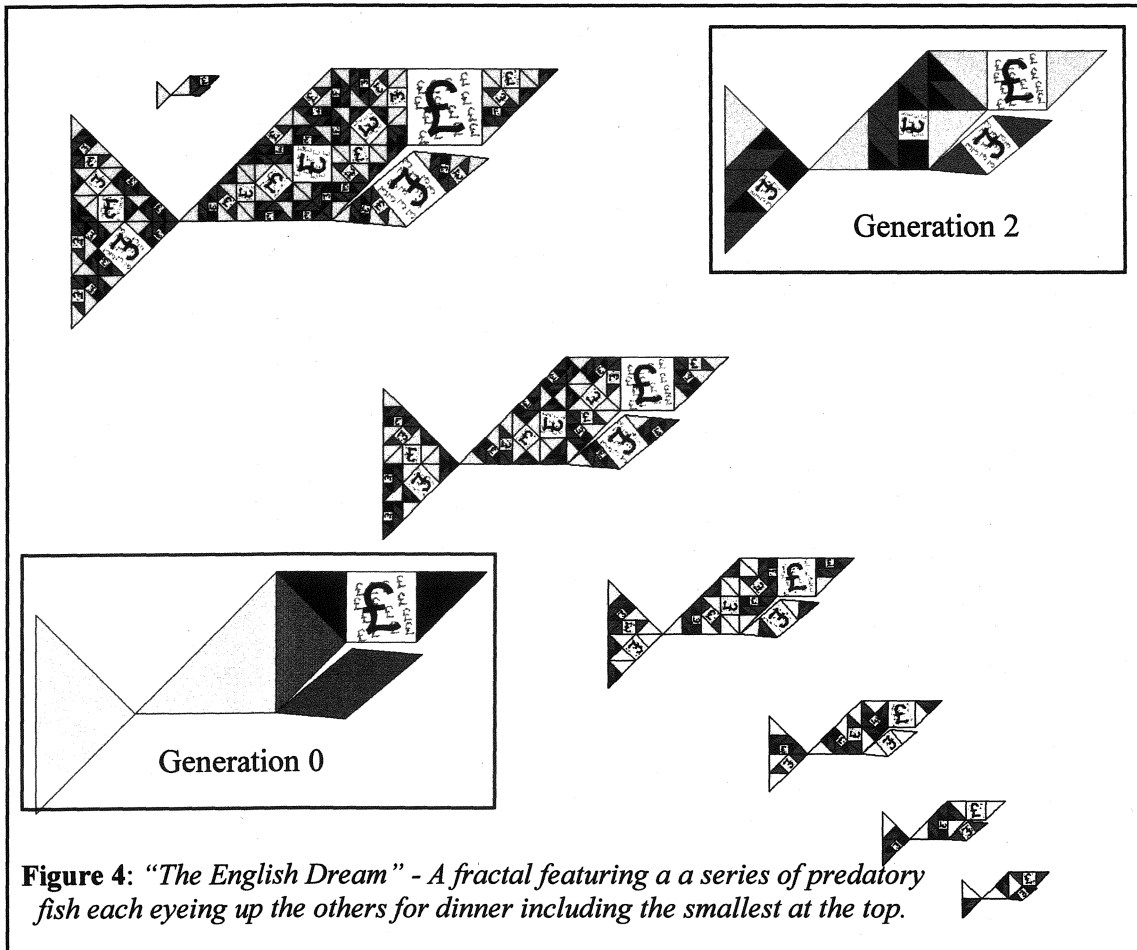


Figure 4: "The English Dream" - A fractal featuring a series of predatory fish each eyeing up the others for dinner including the smallest at the top.

7 Mosaics

The third type is the creation of a mosaic from the design. An example of this is a series of 13 mosaics I have called "Scenes from a Forgotten Hamlet". As a starting point I used one of the 13 convex polygons [8] that can be made from the seven Tangram tiles. These mosaics are designed to cover an A0 size area.

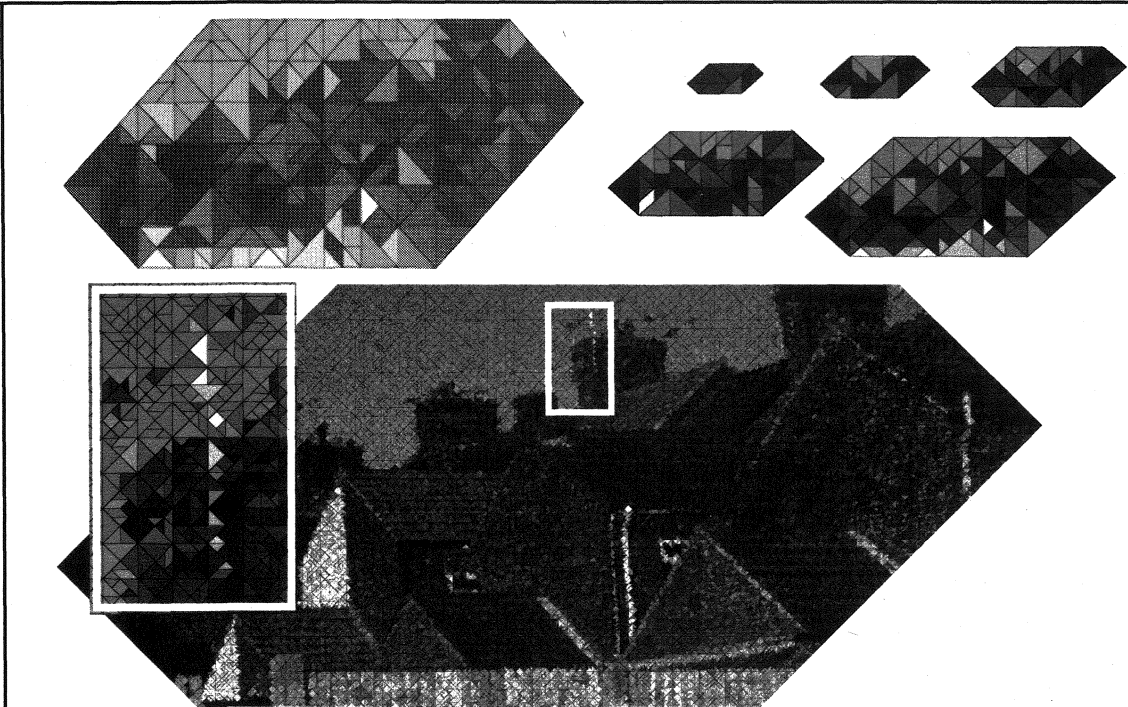


Figure 5: "The Backings" - A group of mosaics from the series "Scenes from a forgotten Hamlet" which are designed to fit A0 (The large white rectangle is an enlargement of the small rectangle)

8 Artwork in Three Dimensions

I have used a method that I have called Tilation. Tilation is the placing of tiles within the boundaries of a polygon, but not necessarily in the plane of the polygon. This is a similar process to stellation. Figure 6 shows the tilation of equilateral triangles. The ones on the left are triangular based pyramids, the ones on the right form a concave arrangement and don't fully cover the triangle, that is, there is a hole in the middle. Each tilation can have concave and convex forms. Once we have a three dimensional tilation we can develop other generations using the methods developed in two dimensions.

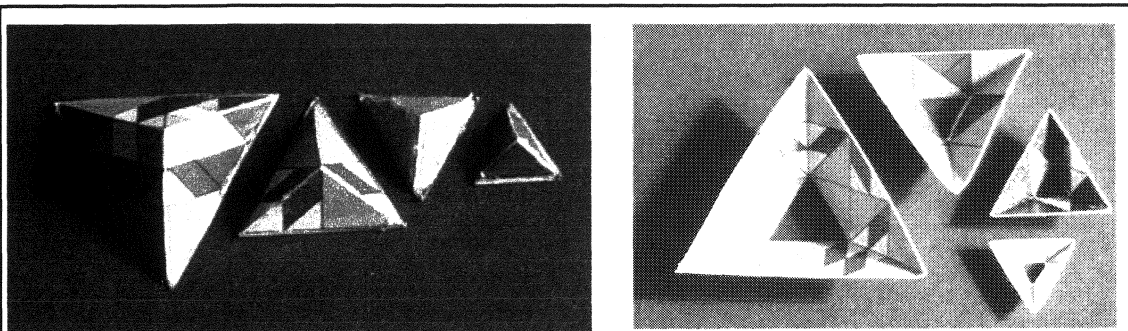


Figure 6: Two ways to tilate an equilateral triangle, each showing 4 generations.

9. General Properties of Three Dimensional Designs

The linear dimensions of each successive generation is $\sqrt{2}$ times larger than the previous generation. The surface area of each successive design is, therefore, double that of the previous generation. If they are constructed from the same material then each one is also double the weight of the previous generation. It can be shown [4] that the weight of the largest approaches the weight of the sum of the previous generations as the number of generations increases.

10. Examples Using the Equilateral Triangle

Figures 7 and 8 illustrate the regular growth patterns for successive generations of tiled equilateral triangles, each being $\sqrt{2}$ times larger than the one previous.

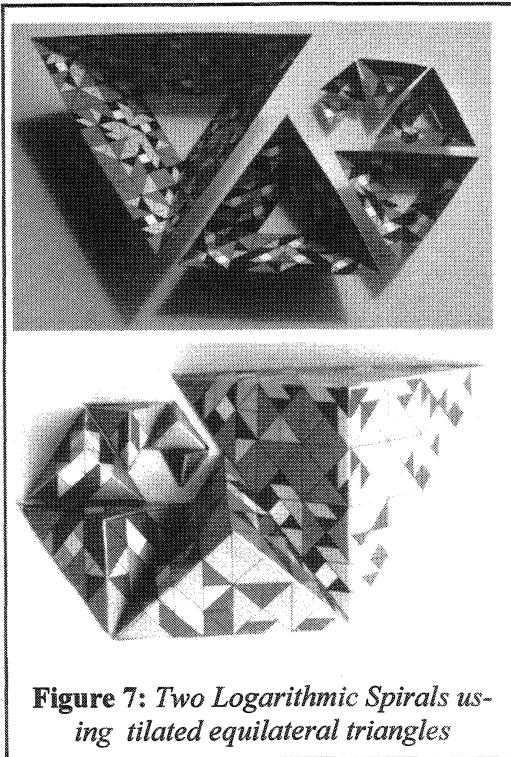


Figure 7: *Two Logarithmic Spirals using tiled equilateral triangles*

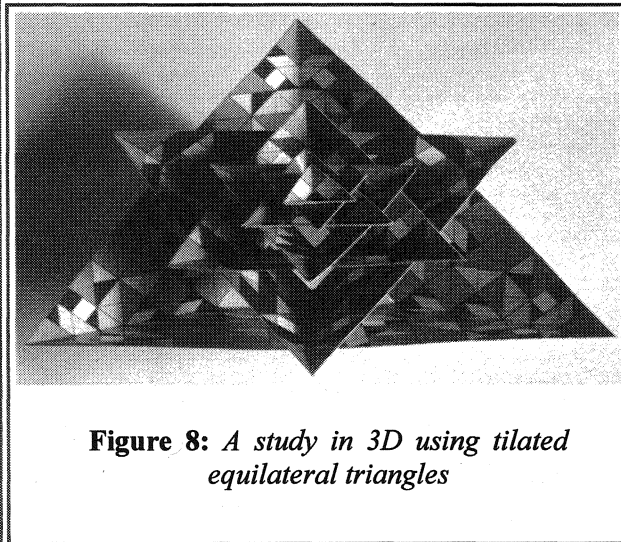


Figure 8: *A study in 3D using tiled equilateral triangles*

11. Tilation of the Square and Pentagon.

There are many ways of tiling the square and pentagon. Those illustrated here are attractive in their own right.

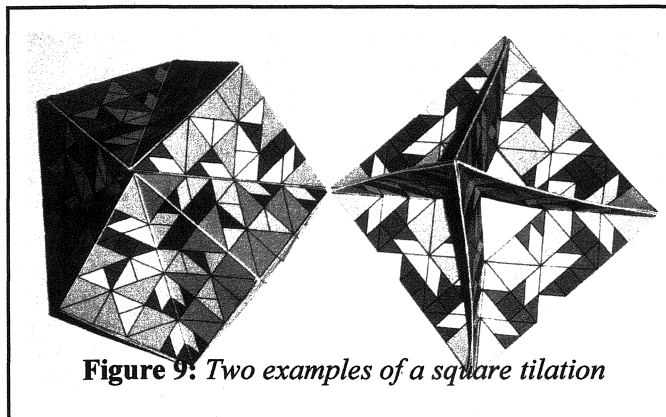


Figure 9: *Two examples of a square tilation*

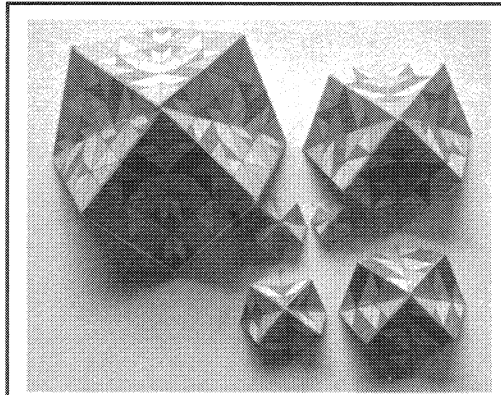


Figure 10: *The first six generations of square tilation*

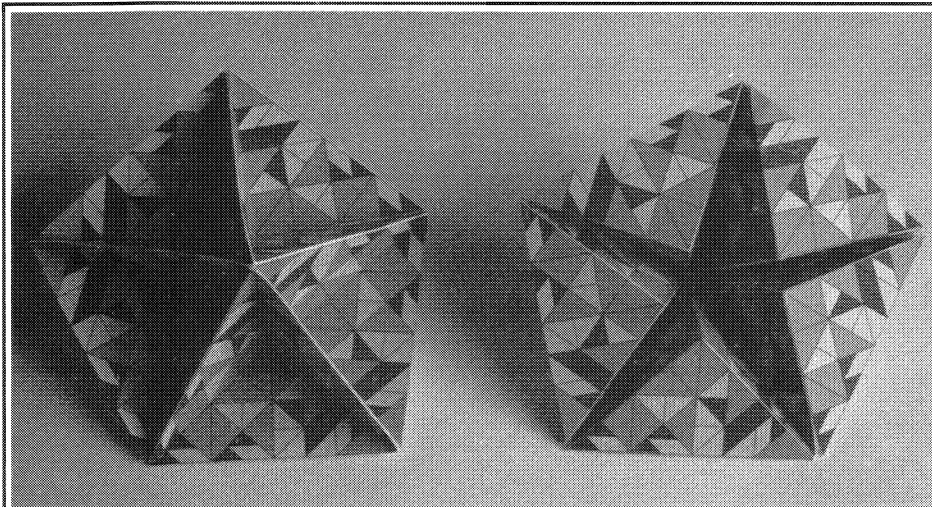


Figure 11: *Two examples of a pentagon tilation,*

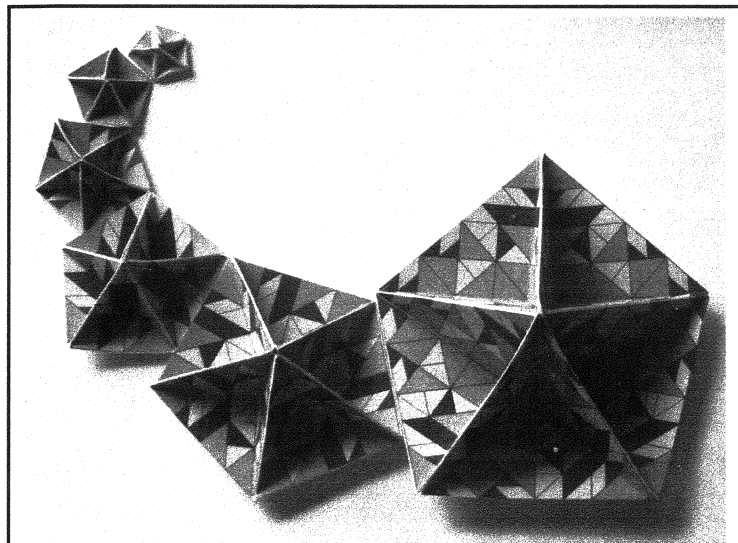


Figure 12: *The first six generations of a pentagon tilation*

11 Tilation of the Platonic Solids

The tilation of the equilateral triangle, the square and the pentagon allows the tilation of the five platonic solids

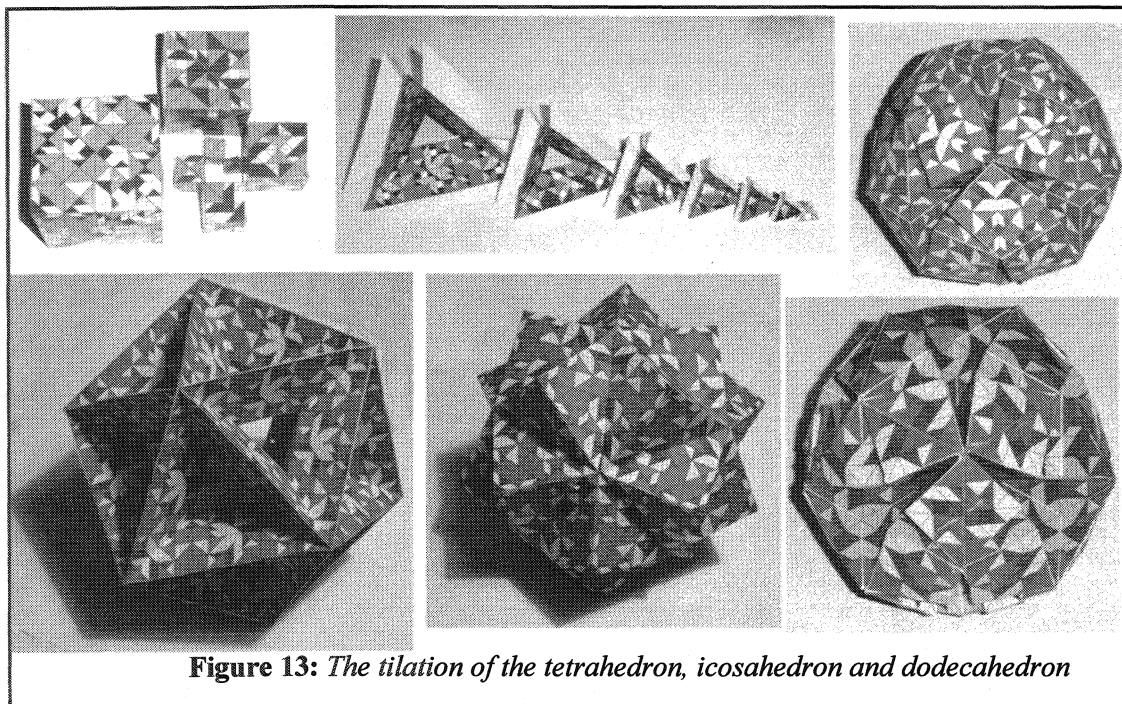


Figure 13: *The tilation of the tetrahedron, icosahedron and dodecahedron*

12 Tilation for Other Objects.

We saw, in section 9, that the weight of each generation of design is double that of the previous one. This leads to the weight of each generation getting closer to the total weight of all previous generations. Figure 14 shows a mobile where each generation balances all the previous ones. The mobile was based upon a six pointed star. The largest star was 600mm. across.

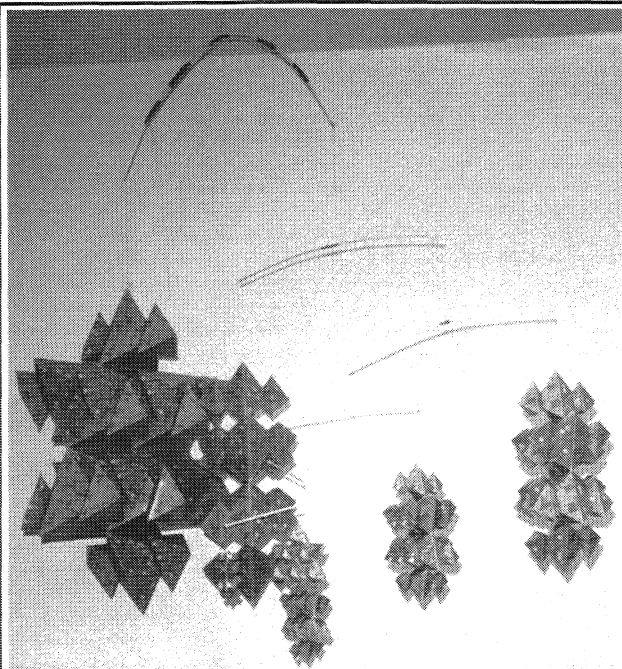
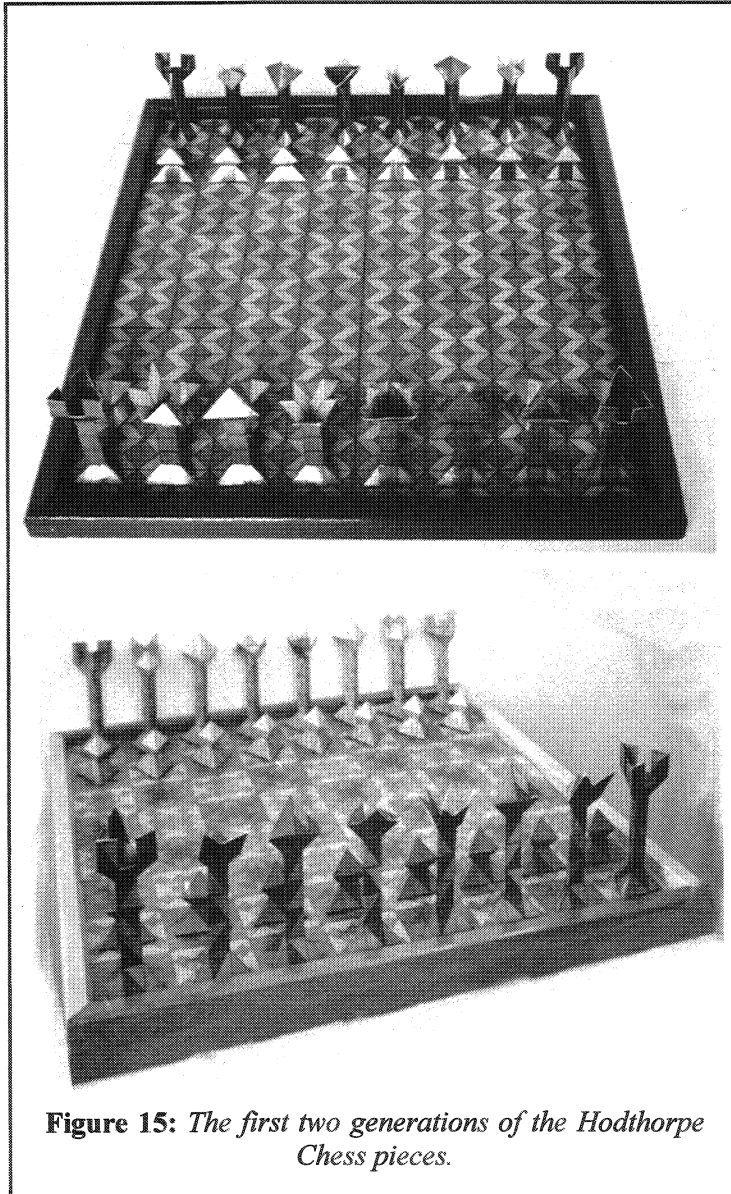


Figure 14: *The series of tiled stars formed into a mobile which illustrates that each is double the weight of the previous generation.*

13 The Hodthorpe Chess Pieces



The tilation of a shape is useful since we end up with a number of 3D elements that have a common interface. This allows them to be fitted together to form more complex solids. This is a bit like the docking of various parts of the space station which rely upon a defined interface to connect them. As an example of this figure 15 shows two sets of the Hodthorpe Chess Pieces. These are based upon the tilation of the equilateral triangle. From the first design an infinite number of designs can be generated, each being $\sqrt{2}$ times larger than the previous one.

Figure 15: *The first two generations of the Hodthorpe Chess pieces.*

14. References

- [1] J J O'Connor and E F Robertson (Accessed 6.12.2003) *babylonian Mathematics* <http://www-history.ac.uk/history/HistTopics/Babylonian_Pythagoras.html>
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