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Measuring the Musical Vocabulary of the Traditional Nzakara Harpists

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There are few musicians left who can play the traditional harp patterns of the Nzakara people of Central Africa. Building upon the musical research of Eric de Dampierre and others, Marc Chemillier has recently provided mathematical analysis of the few harp patterns still played. The harp tunes are made up of two parallel melodic lines, the lower line different from the upper by only a pitch translation and a time delay. Chemillier, however, found that not only do the two melodic lines share this canon-like relationship, but within the harpists' pieces is a cyclical ladder property as well [1]. It is this property which we discuss here.

The traditional harps have 5 strings. The strings are played in pairs, which Chemillier termed "bichords" and which we will soon view as the letters in the alphabet of the Nzakara harpists. There are five allowable bichords: those consisting of neighboring strings are disallowed, as is the pair of the lowest and highest strings. Each harp tune is a succession of bichords repeated over and over again to accompany chant and dance. The smallest repeated set of bichords is the "core" of the harp tune. Within each core, the same letter never appears twice in succession, and the core does not begin and and with the same letter (to avoid repeating the letter upon concatenation). Further, since we choose the smallest repeated set, the core will not "factor" into repeated smaller words. Chemillier encoded the five allowable bichords into $\mathbb{Z}_5$, ordering them from lowest-pitched to highest by the lower note of the bichord and using the upper note to break ties. He found that each allowable core is made up of a generator followed by four successive translations of the generator modulo 5. Since for any translation in $\mathbb{Z}_5$ the pattern returns to the beginning of the core at the fifth translation, each harp tune is a never-ending cyclical "ladder" created by perpetually translating a word in $\mathbb{Z}_5$. The question arises: how many such patterns are there among which the original harpists chose?

For our discussion, we define the following notation and terminology:

$\mathbb{Z}_d$	$\mathbb{Z}_d = \{0, 1, ..., d-1\}$ is our alphabet of symbols.
W	A word of length L
$\mathcal{N}_L$	The set of words of length L containing no $...aa...$ and not equal to $a...a$ for any $a \in \mathbb{Z}_d$. N_L is the number of elements of $\mathcal{N}_L$.
$\mathcal{M}_L$	The set of words of length L containing no $...aa...$ for any a but equal to $b...b$ for some b in $\mathbb{Z}_d$. (That is, $\mathcal{M}_L$ is the set of words beginning and ending with the same letter having no pair of repeated letters in between.) M_L is the number of elements of $\mathcal{M}_L$.
factorable	W can be factored if $W = BB...B$ for some subword B of length less than L.
translation	$B + t$ is a translation of B by t modulo d: every letter in B is translated by t modulo d in the ordered alphabet $\mathbb{Z}_d$.
translation-factorable	W can be translation-factored if $W = B(B+t)(B+2t)...(B+kt)$ for some subword B of length less than L, and for some t and k.

$\mathcal{F}_L$ The set of words of length L that cannot be factored. F_L denotes the number of elements of $\mathcal{F}_L$.
$\mathcal{T}_L$ The subset of $\mathcal{F}_L$ consisting of words which are "evenly" translation-factorable. That is, $W = B(B+t)(B+2t)...(B+kt)$ and $B+(k+1)t = B$. T_L denotes the number of elements of $\mathcal{T}_L$.

Just as not all combinations of letters make actual English words, not all d^L possible words of length L are in the harpists' lexicon. T_L is the number of valid cores of length L in the vocabulary of the Nzakara harpists. We show that N_L and F_L both grow exponentially as L increases. T_L, however, is substantially smaller.

Proposition 1. *For $d \geq 3$ and $L \geq 3$, $N_L = (d-1)^L + (-1)^L(d-1)$.*

Proposition 2. *For $d \geq 3$ and $L \geq 3$, $F_L = N_L - \sum_{m|L} F_m$.*

Corollary. *For every prime p, $F_{p^k} = N_{p^k} - N_{p^{k-1}}$.*

T_L is more difficult to calculate than N_L or F_L. Clearly, $T_L = 0$ if $\gcd(d, L) = 1$. One method of finding T_L could be to view the elements of $\mathcal{N}_k$ and $\mathcal{M}_k$ as generators of $\mathcal{T}_L$.

Lemma. *If W is translation-factorable by subwords of length m_1 and m_2, $m_2 > m_1$, then W is translation-factorable by a subword of length $m_3 = m_2 - m_1$.*

Letting $k = (L \cdot \gcd(d,t))/d$ and $t \in \mathbb{Z}_d$, $t \neq 0$, the lemma allows for a recursive expression for the number of words in $\mathcal{T}_L$ with translation by t in terms of N_k, M_k, and $\sum_{m|L} T_m$. An expected consequence would be that $T_L = (d-2)N_{L/d} + (d-1)M_{L/d} - \sum_{m|L} T_m$ for any prime d. Considering that each word is heard in repeated succession, there are only $T_L/(d \cdot (L/d)) = T_L/L$ words which create a unique pattern when repeated. This is a much more manageable set for study than we were first faced with. These considerations lead to the following questions:

- Is there a special property of $\mathbb{Z}_5$ that led to the harpists' development of this tune structure?
- Do Chemillier's other observations, such as the melodic canon, hold for each of these words?
- How thoroughly did the original Nzakara harpists explore this pattern? What proportion of these cores did they actually use? On what basis did they choose particular cores?

References

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[2] M. Chemillier, 'La musique de la harpe', E. de Dampierre (ed.), *Une esthétique perdue*, Presses de l'Ecole Normale Supérieure, Paris, 1995, pp. 99-208.

[3] *Central African Républic. Music of the former Bandia courts*, recordings, texts, and photographs by M. Chemillier & E. de Dampierre, CNRS/Musée de l'Homme, Le Chant du Monde, CNR 2741009, Paris, 1996.