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Mathematical Connections
in Art, Music, and Science

Fractal Tessellations from Proofs of the Pythagorean Theorem

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Abstract

The Pythagorean theorem can be proven geometrically through the use of dissections of squares and triangles. Four of these decompositions are paired into two compound dissections, which are used to create novel images.

Pythagorean Theorem

Many proofs of the Pythagorean theorem are given in the literature, for example, at the MathWorld web site [1]. Of particular interest to this work are dissection proofs, wherein a figure is cut into pieces and reassembled into another figure with equal area. Several of these are collected at the Cut The Knot web site [2].

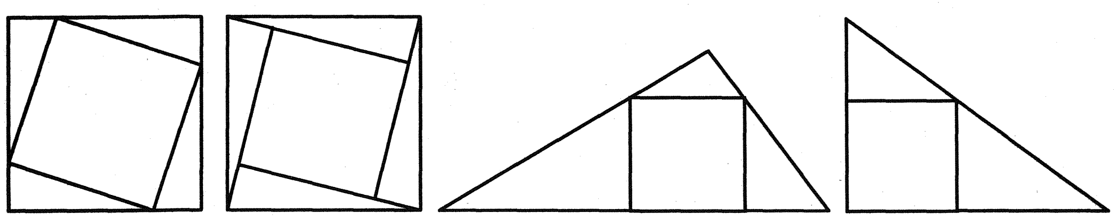
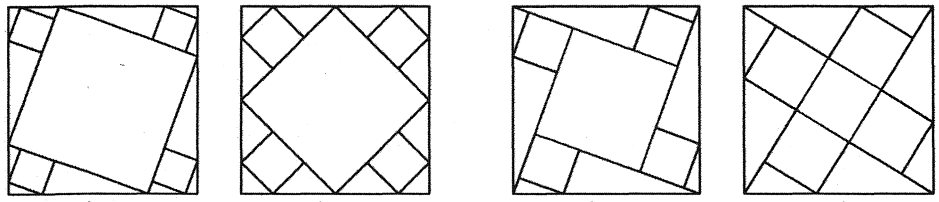


Figure 1: Dissection Proofs

In this work, four dissections were used, as shown in Figure 1. Two were based on decomposing a square into a smaller square and four congruent right triangles. The other two decomposed a right triangle into a square and smaller similar right triangles. In each case, algebraically equating the area of the larger figure with the sum of the areas of the smaller shapes leads to the Pythagorean theorem.

Fractal Tessellations

When dissection results in pieces that can be further dissected, a fractal tessellation can be created by infinitely continuing the process. Fathauer has extensively studied fractal tessellations resulting from dissections; see his "Encyclopedia of Fractal Tilings" [3]. In the present work, each dissection produced both squares and right triangles, which could be further broken down. Each square decomposition was paired with a triangle dissection into a compound process; they are shown in Figure 2. The first two panels show the first square and first triangular dissections paired. The rotation of the inside square is a free parameter, and the two cases correspond to 20 and 45 degrees, respectively. As the rotation increases, the size of the outer smaller squares increases, to a maximum at 45 degrees. The last two panels show the second square and second triangular dissections paired. The rotation angles are 20 degrees and approximately 32 degrees. In the last panel, the rotation was chosen such that the central and outer small squares are congruent.



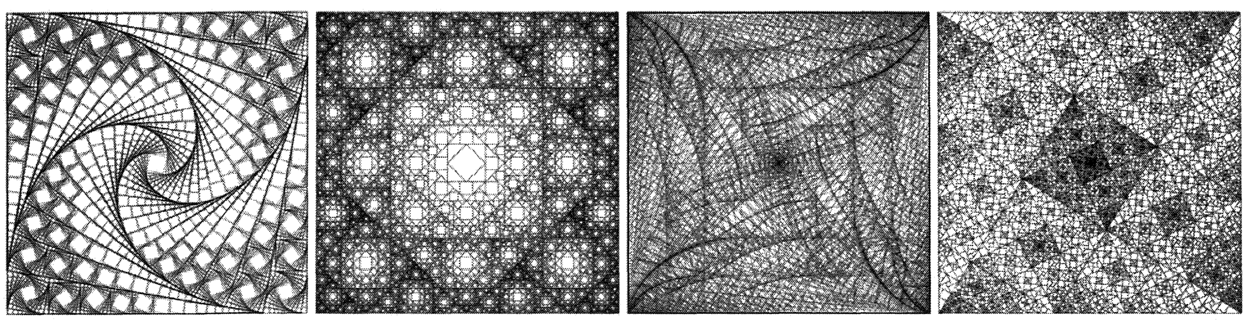
20 degrees 45 degrees
Compound Dissection 1

20 degrees ~31.7 degrees
Compound Dissection 2

Figure 2: Compound Dissections

New Designs

These compound dissections were used to create new artworks, some of which are shown in Figure 3. All were generated using the program Ultra Fractal [3]. These images merely hint at some of the possibilities open to the algorithmic artist. Of course, there are other ways to decompose squares and triangles into more squares and triangles, and each of those can be incorporated into a fractal tessellation. The varieties offered, combined with the freedom afforded by rotating the central square, make this technique quite a fruitful one.



Compound 1
20 degrees rotation
20 iterations

Compound 1
45 degrees
7 iterations

Compound 2
20 different rotations
4 iterations

Compound 2
~31.7 degrees
7 iterations

Figure 3: New Designs from Compound Dissections

References

[1] E. W. Weisstein, "Pythagorean Theorem," http://mathworld.wolfram.com/PythagoreanTheorem.html, 1999.

[2] A. Bogomolny, "Pythagorean Theorem," http://www.cut-the-knot.org/pythagoras/index.shtml, 1996.

[3] R. Fathauer, "Dr. Fathauer's Encyclopedia of Fractal Tilings," http://members.cox.net/fractalenc/encyclopedia.html, 2000.

[4] F. Slijkerman, "Ultra Fractal," http://www.ultrafractal.com, 1997.