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Mathematical Connections
in Art, Music, and Science

Turning a Snowball Inside Out:
Mathematical Visualization at the 12-foot Scale

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Abstract

At the 14th International Snow Sculpting Championships in Breckenridge, CO, Jan 27–31, 2004, Team Minnesota, USA, consisting of the five authors, carved a 12-foot tall representation of the Morin surface – the halfway point of a classical sphere eversion process. This paper describes the design and realization of this ambitious piece of mathematical visualization.

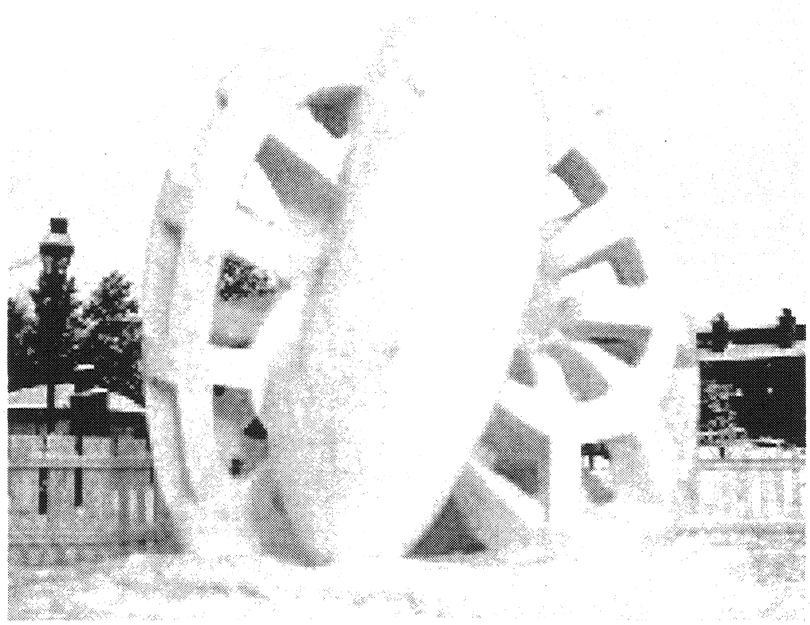

Figure 1. *"Turning a Snowball Inside Out," Snow Sculpting Championships, Breckenridge, 2004.*

1. Introduction

Every year since 1999, Team USA–Minnesota, under the leadership of Stan Wagon [13] has been sculpting abstract geometrical shapes at the renowned International Snow Sculpting Championships in Breckenridge, Colorado. This competition has been held each January for 14 consecutive years. From several dozen submissions, 14 entries were selected for the actual contest, which stretches over five days. The starting material is a 12-foot tall block of hard compact snow with a 10-by10-foot base, weighing more than 20 metric tons. More background on this competition and on Team USA–Minnesota can be found in last year's paper [1].

While celebrating the success of the "Whirled White Web" [6], the team's silver-medal winning entry at the 2003 competition, Carlo Séquin got struck with the idea – which later turned into an obsession – of showing the general public a way of "Turning a Snowball Inside Out." Thinking about previously built models of the Boy surface and the Morin surface, which depict halfway points in different sphere eversion processes, he proposed to build such a model as a giant snow sculpture. To make this dream a reality, several constraints had to be met:

- A representation had to be chosen that shows, in a single static frame, as much as possible of the relevant geometry of the halfway inside-out sphere.
- The shape had to be transformed to fit the given snow block dimensions: 10 × 10 × 12 feet tall.
- This resulting shape had to show some aesthetic qualities as an abstract sculpture.
- And, last but not least, the shape had to be constructible in snow in a limited amount of time.

This paper focuses on these modeling issues and gives a brief description of the actual construction.

2. Design Evolution

Several factors favored the Morin surface [5] over the Boy surface for this particular visualization of the sphere eversion process. The 4-fold symmetry of the former naturally fits into the square cross section of the snow blocks at the Breckenridge championships. No coloring of the snow is allowed; so in order to differentiate the two different sides of the surface, some kind of different texturing has to be employed. For the Morin surface, this approach readily assigns the same texturing to opposite lobes. For the Boy surface, this technique cannot be used so easily, because one would need to show somehow a double covering of that surface to depict the halfway point of a true sphere eversion.

Many different shapes of the Morin surface have been depicted in the literature and in educational video sequences. Charles Pugh and Nelson Max, whose classical video [4] has just been re-released, used a geometrical form resembling a twisted Mexican hat (Fig.2a) to morph the surface through the various topological events into a 4-fold symmetrical halfway point. Denner and Apéry [2] showed how a cuboctahedron (with its square faces suitably split into two triangles) can be turned inside out using a regular homotopy (i.e., without ever letting the dihedral angle along any edge collapse into a 0-degree knife edge). The resulting halfway point is shown in Figure 2b. John Sullivan subjected this shape to Brakke's *Surface Evolver* program, where it was suitably subdivided and smoothed to minimize its Willmore bending energy [15][3], while maintaining its 4-fold symmetry. After a slight disturbance of that symmetry, the Surface Evolver will then let this shape veer off this saddle point in Willmore energy space and slide down into the absolute energy minimum of a perfectly round sphere. A beautiful movie, premiered in 1998, shows this process [10]. The minimum energy halfway point looks roughly like a symmetrical constellation of mutually intersecting bubbles (Fig.2c). While aesthetically very pleasing, it does not easily convey the mutual connectivity of these bubbles – particularly not when the surfaces are made completely opaque.

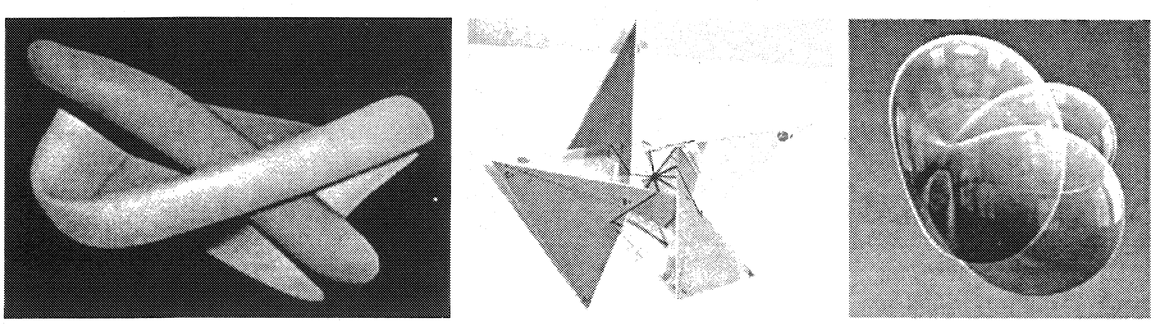

Figure 2. *Different representations of the Morin Surface: (a) Max, (b) Denner, (c) Sullivan.*

The Morin surface basically has four "ears" that look like the "mouth" of the classical depiction of a Klein bottle, where a tubular portion of the surface turns inwards and passes through another, flatter part of the surface, thereby forming some "ear drum" within that tube. To make the geometry of that surface as understandable as possible, it is necessary to make these eardrums visible. In the minimum-energy version of the Morin surface, the ears are closed, and the eardrums are not visible; as the sphere moves through the halfway point of the eversion, two ears are just starting to open up while the other two are closing.

If we take Denner's polyhedron (Fig.2b) and simply run it through a subdivision algorithm, we obtain a result as shown in Figure 3a. In this model the ears are badly formed. To obtain better shape control, we created a 4-fold symmetrical, parameterized subdivision surface, defined by 30 vertices with a total of 22 degrees of freedom, and we played with these parameters until we had a pleasing shape that displayed the eardrums nicely (Fig.3b). We then scaled that surface further, so that it would fit nicely into the given snow block dimensions; a prototype made on our Zcorp 3D printer is shown in Figure 3c.

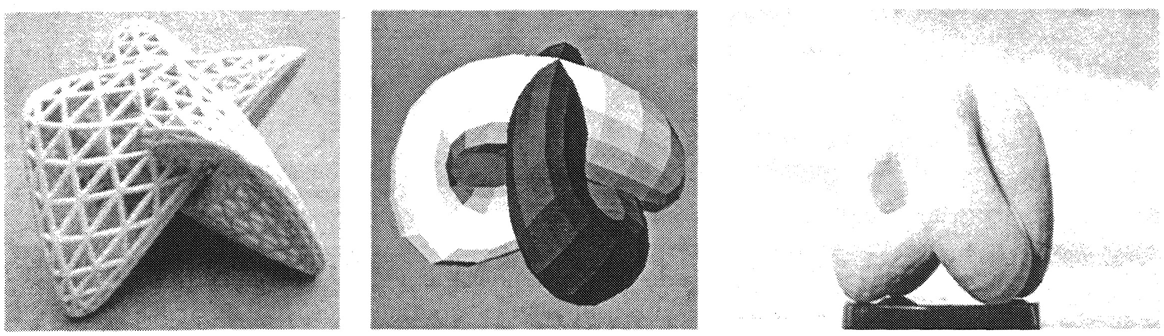

Figure 3. *Adjusting the Morin Surface: (a) cuboctahedral, (b) parameterized, (c) fit to block.*

In the Breckenridge competition one is not allowed to color the snow as suggested by Figure 3b. Instead, we might have used two different surface textures – a smooth, polished finish to represent one color, and a chiseled texture for the other one. However, we felt that the resulting surface neither had enough artistic drama, nor was particularly interesting or challenging to carve. Moreover, it would be nice if the spectators could see how the end of one curved lobe smoothly merges into the central ear of the next lobe. To show this, some degree of transparency of the surface is required. Inspired by the prize winning Swiss sculpture of 2001, "Circle of Life", we decided to adopt a gridded strut system to give the surface some transparency. Now the parameterization of the surface becomes important, since it implicitly defines the positions of all the struts. A gridded surface representation is shown in Figure 3a; but it is clear that it would be impossible to hollow out the inner parts of the lobes, if the holes between the struts are too small. Thus the number of struts has to be reduced dramatically.

If a highly curved surface has to be depicted by a minimal number of struts, the placement and shape of these struts becomes extremely important. Using our subdivision approximation of the Morin surface, we were not able to obtain a satisfactory strut pattern. The subdivision process is too automatic and does not give us enough control over the directions and exact placements of the struts, and thus over the sizes and shapes of the holes between them. For instance, to keep the sculpting task simple and to maximize stability, we decided to outline each eardrum along the self-intersection line of the surface with a peripheral strut.

Thus we resorted to a different approach. We modeled the lobes of the Morin surface as sweeps, in which both the cross section of the sweep and the sweep path itself were modeled as cubic Bézier curves. Fortunately, SLIDE [8], our own experimental CAD modeling system at U.C.Berkeley, has a powerful sweep operator that allows the cross section to be morphed as it moves along the sweep path. At every control point of the sweep curve, the cross section can be non-uniformly scaled and rotated around all three coordinate axes. The values of these five control parameters are then blended with the same cubic polynomial as the positional coordinates of the sweep path. This gives us enough degrees of freedom to define pleasing-looking lobes, where the cross section at the end of the sweep exactly matches the sweep path of the next lobe, in shape as well as in orientation, thus forming a perfect ear (Fig.4b).

The SLIDE software also provides parameters that control the fineness of the piecewise linear approximation of any curve or surface. A simple gridding utility allows the user to carve out the central portion of each facet, thus leaving only a grid-like structure to depict the surface. With these facilities we were able to construct computer models of possible shapes for our proposed sculpture and to construct physical prototypes on one of our rapid prototyping machines.

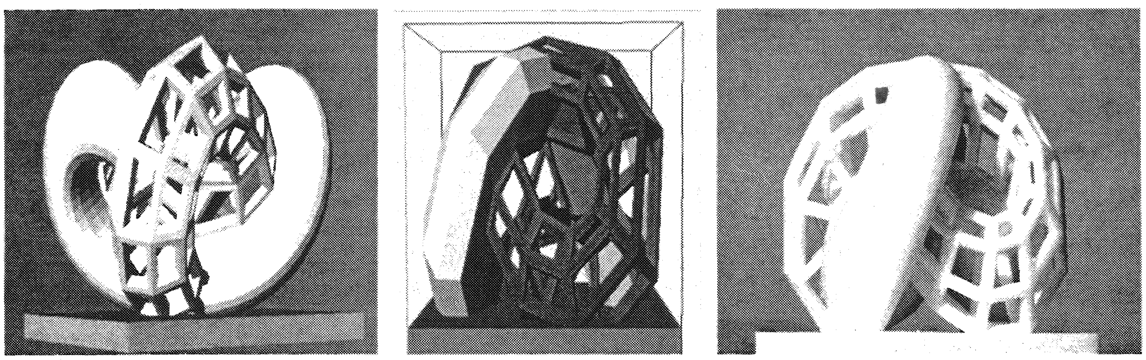

Figure 4. *Different orientations of the Morin Surface: (a) flower-like, (b) (c) arch- or dome-like.*

While maintaining 4-fold symmetry around the vertical axis, we still had the option of orienting the Morin surface with the four lobes pointing upwards, forming a flower-like structure (Fig.4a), or standing the surface on these four lobes, forming a more dome-like structure (Fig.4b,c). We chose the latter approach because our team was still reeling from the emotional scar of having our 2003 snow-sculpture collapse 45 minutes after judging had ended. We thus decided that opposite lobes of our sculpture should form arches that can easily support their weight, rather than a U-shape (Fig.4a) that might be torn apart by gravity. In hindsight, with the weather as nice and cold as it was in 2004, we could probably have built the flower configuration and then would have ended up with an even more dramatic looking sculpture.

Next we had to decide how exactly to grid the surface. We decided to apply the gridding to only two of the four lobes. First, this would reduce the amount of carving we had to do, and would also increase the stability and strength of our sculpture. Moreover, the gridded lobes in contrast with the smooth surfaces would be the differentiating quality for representing inside and outside of the original spherical shell (Fig. 4). Conceptually, we might pretend that the sphere surface was made of some kind of one-way mirror glass. From one side this material would be completely opaque and display a smooth surface; from the other side the material would be transparent and show itself as a grid structure. This analogy breaks down when one looks into one of the gridded lobes and then sees the inside of the back wall, which from this direction should now be opaque. John Sullivan has a rendering program that depicts this one-way mirror quality perfectly [11].

This state of modeling (Fig.4c) was good enough to draw the required hand-drawn sketch needed for our submission in July 2003 (Fig.5a). However, it was clear, that the linear struts were not optimal to represent the smooth shape of the lobes. Alex Kozlowski enhanced our SLIDE sweep utility with a couple of extra parameters that allow the surface to be tessellated at a much finer level than indicated by the number of struts. This enhanced program can produce strut surfaces that are curved in both dimensions. A further enhancement allowed us to place the struts not just at the strict intervals defined by the parameterization of the sweep path, but to place them at a finer level of discretization. With all these enhancements, we could make a model on our 3D-Printer [14] that matched our sketch closely and showed what we wanted to achieve during the snow sculpting competition (Fig.5b).

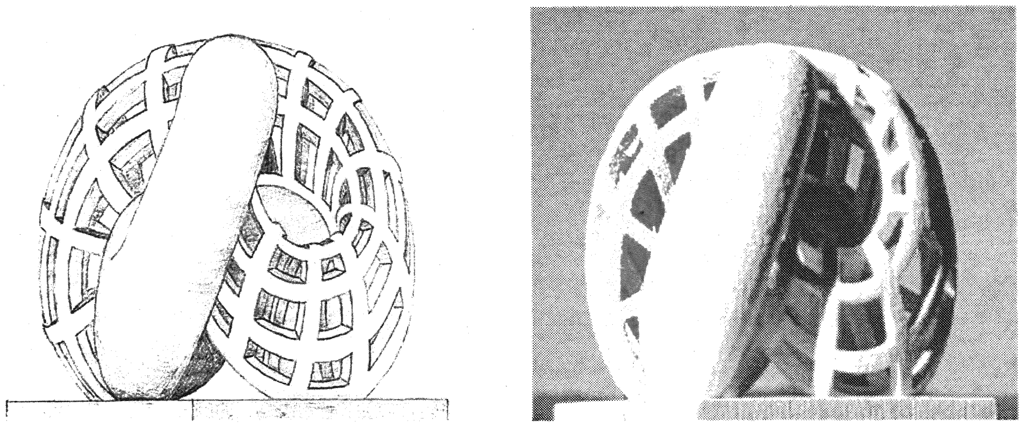

Figure 5. *Proposed sculpture: (a) sketch for submission, (b) refined model.*

After we heard that we had been accepted to the championships, we also employed our Fused Deposition Modeling (FDM) machine from Stratasys [9] to build two more sturdy green models for use at the sculpting site (Fig.6a,b). The model with the solid lobes represents an intermediate stage of construction. This model came in handy when we decided on Day 3 to slightly simplify the lattice structure.

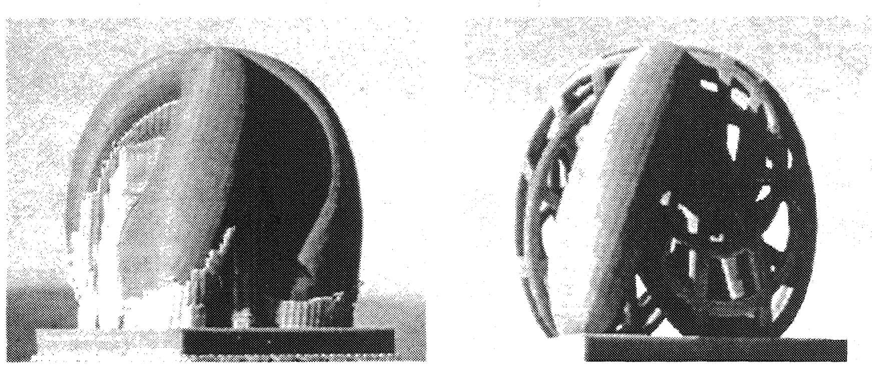

Figure 6. *Maquettes made on the FDM machine for use at the sculpting site.*

3. Practice Run

We had the opportunity to try out our ideas two days before the competition on a 40% scale practice snow block, prepared by Stan Wagon in the backyard of his house in Silverthorne, Colorado. Our goal was to find a good way to get started and to remove a massive amount of snow from the block on the first day. We also wanted to try out a novel system to mark the struts.

Because of the slanted, spirally shape of the lobes, it was difficult to figure out what parts of the snow block could be safely cut away. We tried to use the top profile (Fig.7a) and drill down to the level of the saddles in front of the ears to remove significant amounts of snow. In addition we tried to also extrude upwards from the pedestal the outermost projection of the four lobes (Fig.7c). That first phase seemed to go reasonably well (Fig.8a,b). Trouble started when we tried to hollow out two of the ears and to connect them into the gridded tubes emerging on the other side. We missed this connection and created unwanted holes and tunnels.

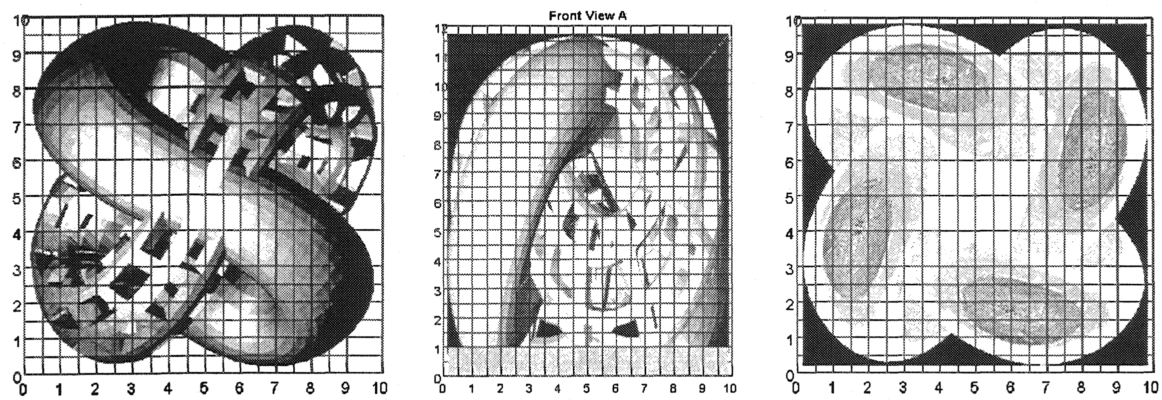

Figure 7. *Computer projections with dimensions: (a) top, (b) side, (c) top of base section.*

We also ran into difficulties when carving the struts. We could define them reasonably well by nailing 4″-wide cloth strips to the surface of the lobes. But at the 40% scale, the holes were two narrow to remove the snow from inside the lobes efficiently. Thus the work on the practice block revealed the difficulty of our undertaking and showed where the most likely causes of errors might come from.

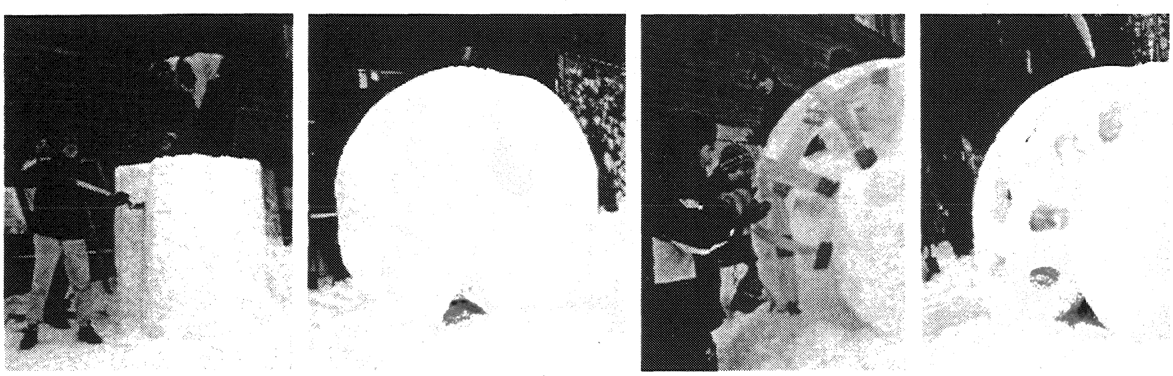

Figure 8. *Various stages of sculpting the 40-percent-scale practice block.*

4. Actual Implementation

Competitive sculpting on the large blocks started on Tuesday 1/27/04 at 11 am. We had changed our opening moves slightly. On our model we had determined the position of two planes that would define the wedge of maximal volume between two subsequent lobes. We marked these planes on the block, and then we drilled a sequence of closely spaced holes along these planes into the wedge. This allowed us to remove some reasonably large chunks of snow efficiently. By the end of Day 1 we had removed these wedges on all four sides of the block (Fig.9a).

On Day 2 we sculpted from 8 am until about 9 pm to create four reasonably smooth lobes with an outline of an ear inside (Fig.9b). We were careful not to remove too much material near the ears. To assist us, we had crafted templates out of foam core, one to define the outline of the future strut where the lobe would touch the wedge planes, and one for the shape of the eardrum.

Figure 9. *State of sculpture (a) Tuesday, (b) Wednesday night, (c) Thursday noon.*

All this measuring and marking took more time than expected, and it was Thursday noon before we had achieved a reasonably smooth and symmetrical definition for the four lobes; this was half a day later than we had originally planned. Based on the difficulties experienced with the struts on the practice block, and our delay in schedule, we decided to simplify the lattice slightly compared to the original model. We reduced the number of longitudinal struts from 5 to 4, and the number of horizontal struts from 7 to 5, while widening the holes by several inches over the original plans. On our small plastic model with the four solid lobes (Fig.6a), we determined the optimal strut positions by sticking on narrow strips of insulating tape and moving them around to obtain a pleasing distribution that also promised to be of reasonable stability. For that purpose, we moved many struts into positions where they would solidly run into the base pedestal of the sculpture; being properly anchored there, they would help distribute the stress and carry the weight of the snow above.

To transfer this strut pattern onto two lobes, we used 10″-wide cloth strips, which we draped over the surface, pinning them down with nails and small cardboard pieces (Fig.9c). Laying out the struts that roughly followed geodesic lines on the lobes was quite easy. But the struts with strong geodesic curvature were difficult to layout and required many extra nails to hold them in place. We also had to fight the wind, which tended to flip over the cloth strips. Thus our "novel" marking systems was not an unqualified success.

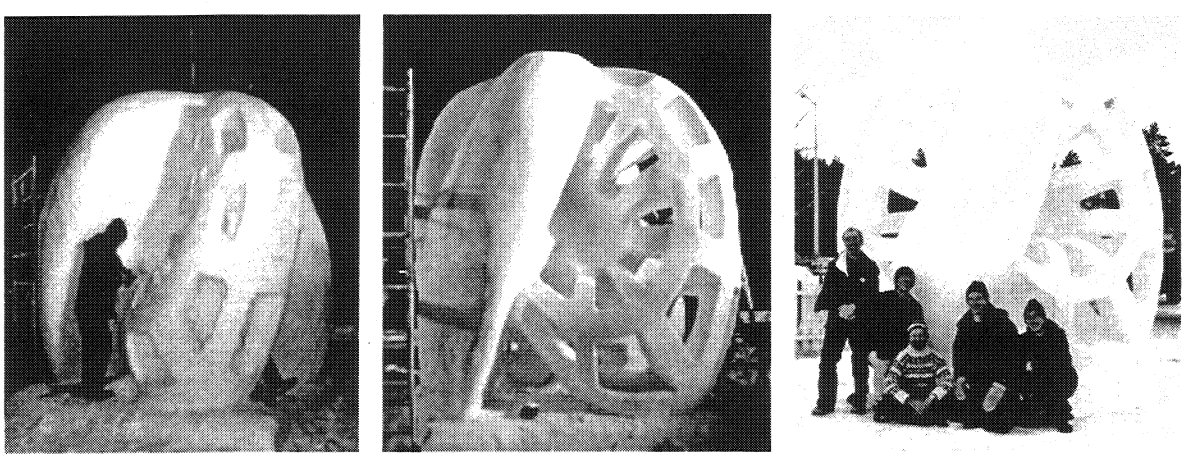

Figure 10. *State of sculpture (a) Thursday night, (b) Friday night, (c) Saturday noon*

We then started to chisel in the quadrilateral and triangular openings between the cloth strips. Gradually we worked our way inwards through all the openings until these tunnels started to join near the middle of the lobes. By Thursday night we had created the proper topology on one of the lobes (Fig.10a). During Friday we refined the geometry of the lattice (Fig.10b) and tried to make the struts on the opposite lobe an exact copy of the first.

Saturday morning we were on site by 6 am and started the final refinement process. Since the weather was reasonably cold, we narrowed the struts by another couple of inches, and tried during this process to make them as uniformly rounded as possible and to align the individual strut segments to one another. With this task we used up all available time (Fig.10c). There was no time to make a particularly smooth surface finish and to pick out all little stones and ice chips; and the surface of the pedestal was only roughly leveled and smoothed. Perhaps this was just as well; within a couple of hours it started to snow, and the little blemishes (as well as many of the details on other sculptures) were gently covered up.

5. Results

We had achieved our main goal to carve a Morin surface with some dramatic and attractive appearance that held up for many days. While we were basically satisfied with our accomplishment, we were well aware that the model did not look as even and smooth as the "Whirled White Web" of the previous year [1][6]. There were several other competing sculptures with much more refined appearances, and some with very original and carefully thought-out designs. Several were arguably superior to ours in their direct aesthetic appeal. Thus we were not surprised when the first three places went to three of those exquisite sculptures. Gold was won by "Paradigm Shift" by Team Canada–British Columbia (Fig.11a); the silver medal, Artists' Choice, as well as People's Choice were given to "Year of the Dragon" by Team USA–Tennessee (Fig.11b), and bronze was won by Team Canada–Ontario with "Winter Oasis" (Fig.11c).

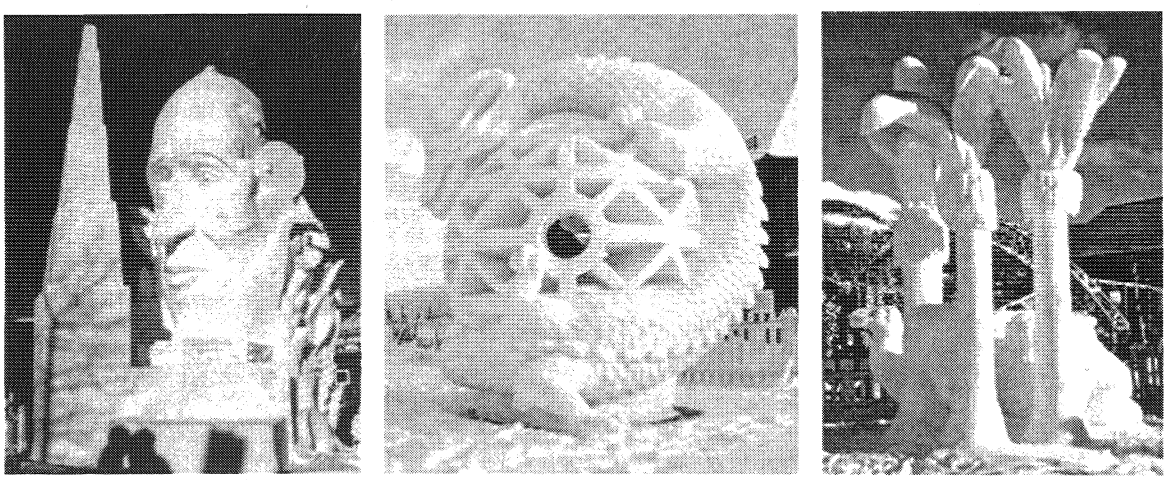

Figure 11. *Winners: (a) Gold: "Paradigm Shift", (b) Silver "Year of the Dragon", (c) Bronze: "Winter Oasis".*

Our sculpture received an honorable mention as the "Most Ambitious Design" (Fig.12a) – a description that made us very happy. Another honorable mention was also given to the team from Germany who sculpted an origami bird breaking out of its confining cardboard box (Fig.12b). There were many other exquisite sculptures, such as "Original Rodeo" (USA–Idaho), winning the title of "Children's Choice", or "Flower Guy" (Bulgaria) and "Museum Piece" (England).

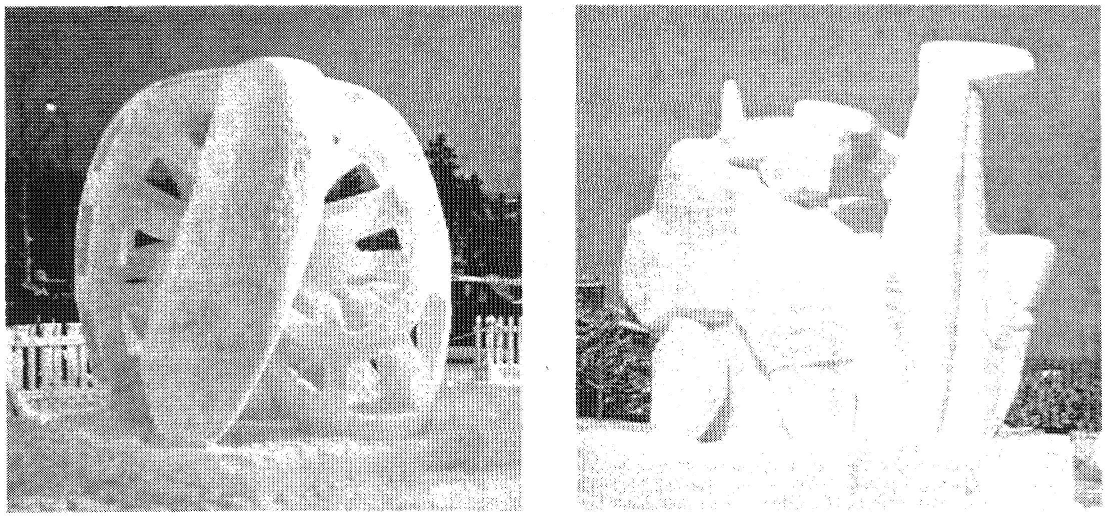

Figure 12. *Honorable Mention: (a) "Turning a Snowball Inside Out", (b) "A Chance".*

Unfortunately, Saturday morning was strongly overcast, and snow started to fall shortly after judging ended. Thus on that day it was impossible to take photographs that showed these sculptures in their full detail and in good light. The sun only came out again the following day, after three inches of snow had fallen. Because of its bold features, our sculpture did not lose much of its appeal with this extra snow coverage (Fig.12a). Some teams returned later that day to brush off their sculptures to reveal their details again, and thus make visible the fruits of 5 days of labor. More photos of these sculptures can be found at [7], [12], [13].

6. Conclusions

Did we get our message across? Hardly! People were definitely intrigued by the title. Many studied the sequence of 14 images that we had prepared to show the process of eversion from a gridded green sphere to the inside-out finish in the form of a solid red ball. But often their final reaction was "I don't get it!" Only when we talked to interested parties and demonstrated the overall goal with a beach ball into which we had cut a hole, did they start to grasp the problem and the rules of the game, and learned to see at least how the process may get started. Still, we feel the exercise was very worthwhile. Several thousand people saw our sculpture, and for most of them this may have been a first intriguing exposure to the counter-intuitive notion that a sphere can be turned inside out. Alex Kozlowski and Carlo Séquin will try again to get this concept across to a larger audience by submitting a whole series of colored rapid prototyping models to the SIGGRAPH art gallery 2004.

For all five authors, this venture led to a deeper understanding of the Morin surface and of the sphere eversion problem – we now can sketch that surface in our sleep. A special thrill for all of us was the opportunity to see this elusive surface at a very physical 12-foot scale, and even climb inside! What medium other than snow could allow one to create an intriguing shape at that scale in only five days? How long would it take to carve that same shape in stone? We all gained a lot of pleasure and satisfaction from seeing our vision emerge so rapidly from an originally rather forbidding looking block of snow. The transitory nature of our creation is the small price we have to pay.

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