

Mathematical Magic

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Abstract

In this paper, we present simple strategies for performing mathematical calculations that appear magical to most audiences. Specifically, we explain how to square large numbers, memorize pi to 100 places and determine the day of the week of any given date.

Squaring Large Numbers

The algebraic identity $A^2 = (A + d)(A - d) + d^2$ can be used to square two-digit numbers in your head, almost as quickly as someone can do on a calculator. With practice, one can even square three-digit numbers (and higher) by this method. Let's illustrate.

Given a two-digit number A , let d be the distance between A and the nearest "easy" number. (I define a number to be "easy" if it only has one nonzero digit.)

For example, to square the number 23, let $d = 3$ and compute

$$23^2 = (26)(20) + 3^2 = 520 + 9 = 529.$$

To square 48, let $d = 2$ and compute

$$48^2 = (50)(46) + 2^2 = 2300 + 4 = 2304.$$

Notice how much easier the above calculations are than a traditional "foil" computation like $48^2 = 40^2 + 8^2 + 2(40)(8)$.

Numbers that end in 5 are especially easy. For example,

$$85^2 = (80)(90) + 5^2 = 7200 + 25 = 7225.$$

Once you can square two-digit numbers quickly, you can attempt the mental squaring of three-digit numbers. For example, suppose you wanted to square 223. Here, we let $A = 223$, and $d = 23$, then compute

$$(223)^2 = (200)(246) + (23)^2 = 49,200 + 529 = 49,729,$$

where $(23)^2$ is calculated as illustrated earlier.

Note: I recommend doing all single-digit multiplication (and almost all other mental computations) from left to right.

To compute $(200)(246)$, first multiply

2 times 200 = 400, followed by
2 times 40 = 80, then add those partial results to get
400+80 = 480, then add
2 times 6 = 12 to get 492,
Appending two zeros gives **49,200**.

With practice, multiplying a single-digit number times a two-digit, three-digit, or four-digit number can also be done in a matter of seconds.

Squaring the number 848 is considerably more challenging. Letting $d = 48$, we see that
 $(848)^2 = (800)(896) + (48)^2$.

We compute 8 times 896 as $6400 + 720 = 7120$, followed by adding 8 times $6 = 48$, to get 7168. Consequently, $(800)(896) = 716,800$. If you can *quickly* compute $(48)^2 = 2304$ by the earlier method, then the problem reduces to $716,800 + 2304 = 719,104$. Here, we perform the addition from left to right: $716,800 + 2000 = 718,800$, then add 304 to get **719,104**.

The problem you will initially encounter is a memory issue. By the time you finish computing $(48)^2$, you might have forgotten 716,800. How do you hold on to that number, and still have room in your brain to compute 48 squared? One way is to turn 716,800 into “COTTAGE FACES” using a simple mnemonic device described in the next section.

Memorizing pi or any other number

Commit to memory the following silly sentence:

“My turtle Pancho will, my love, pick up my new mover Ginger.”

Say the sentence a few times until you know it cold. Got it? Congratulations. You have just memorized the first 24 digits of pi. We are not using the slow and impractical method of associating each digit with the length of each word. Instead we take advantage of a *phonetic code* that has been in the English language for over 100 years. Here it is:

1 = t or d
2 = n
3 = m
4 = r
5 = l
6 = ch, sh, or j
7 = k or hard g
8 = f or v
9 = p or b
0 = s or z

There are even mnemonics for learning this list, but we will let you create your own associations. Now you can turn numbers into words by inserting vowel sounds (or the consonant sounds of h, w, or y) around or in between the consonant sounds in your word.

For example, the number 946 could become: bridge, birch, brush, perish, porridge, or preach, to name a few. Notice that spelling does not matter. Even though porridge is spelled with a double-r,

the r sound only occurs once. Although a number can have many, many different words representing it, *a word can only represent a single number*.

Thus, our original sentence for pi translates back to:

3 1415 926 5 3 58 979 3 2 384 6264

The next 17 digits of pi can be encoded in the following sentence:

“My movie monkey plays in a favorite bucket.”

This codes the next 17 digits of pi, namely

3 38 327 950 2 8841 971

The next 19 digits of pi:

69 3 99 375 1 05820 97494

become **“Ship my puppy Michael to Sullivan’s backrubber.”**

The next 40 digits of pi are:

45 92 307 81 640 62 8 620 and 8 99 86 28 0 3482 5 3421 1 7067

which become **“A really open music video cheers Jenny F. Jones”** and **“Have a baby fish knife, so Marvin will marinate the goose-chick!”**

You can use this method to memorize more practical information, such as phone numbers, birth dates, and more. In the previous section, I recommended that the number 716,800 is easier to store as “cottage faces,” especially while manipulating other numbers. (Alternatively, you might simply turn 716 into “Cottage” and hold the 8 on your fingers. After all, what are hands for, if not for holding onto digits!)

Although I do not use mnemonics when squaring a three-digit number, I do need them for squaring four-digit numbers. For example, to square the number 3223, I compute

$$(3223)^2 = (3000)(3446) + (223)^2.$$

To compute 3 times 3446, I split the four-digit number into groups of length two:

3 times 3400 equals 10,200; 3 times 46 equals 138.

Thus (doing all calculations from left to right), 3 times 3446 equals 10,338 and

$$(3000)(3446) = 10,338,000.$$

At this point, I can actually say (for my audience to hear) “ten million”.

Note: The reason that I know that ten million will not turn into 11 million is that the largest three-digit square that I will ever have to add is $(500)^2 = 250,000$. (Remember, I let d be the distance to the *nearest* thousand.)

Since 338,000 is below 750,000, then I can say “ten million” with complete confidence, and free up two valuable digits of memory space.. To ensure that I do not forget 338,000, I convert 338 into MY MOVIE. This will stand out like a sore thumb while I compute $(223)^2 = 49,729$. I might convert the last three digits, 729, into CANOPY.

Finally, I add 49 (thousand) to MY MOVIE = 338 (thousand) to get 387 thousand, which I promptly say. Finally, converting CANOPY back to 729, I can say the final three digits. Thus, I have given the entire answer **10,387,729**. (Amusingly, it is sometimes the case that by the time I have finished saying the entire answer, I cannot repeat the answer and often cannot even repeat the initial problem!)

Calendar Calculating

Finally, we end with an explanation of a skill that is as practical as it is entertaining: how to determine the day of the week of any date. We first describe the method before discussing its mathematical foundation.

Every **day of the week** is given a code number:

Sunday = 1
Monday = 2
Tuesday = 3
Wednesday = 4
Thursday = 5
Friday = 6
Saturday = 7 or 0

Next we assign a code to each **month**:

January = 1 *
February = 4 *
March = 4

April = 0
May = 2
June = 5

July = 0
August = 3
September = 6

October = 1
November = 4
December = 6

(* On leap years, January has month code of 0 and February has month code of 3)

This list is easy to remember in blocks of three: 144 is twelve squared; 025 is five squared; 036 is six squared; 146 is *almost* twelve squared!

Finally, we compute the **year** code:

For years in the 1900's, the year 1900 + X (where $0 \leq X \leq 99$) has year code
 $X + (X/4) \text{ MOD } 7,$

where we ignore any fractional part when computing $X/4$. In other words, from the number $X + X/4$, we subtract the largest multiple of 7 that we can.

For instance, 1983 has year code: $83 + (83/4) \text{ MOD } 7 = 83 + 20 \text{ MOD } 7 = 103 \text{ MOD } 7 = 103 - 98 = 5$.

Here is the formula for determining the day of the week for any date:

$$(\text{Year code} + \text{Month Code} + \text{date}) \text{ MOD } 7 = \text{code for day of the week}$$

For example, March 20, 1983 has day of the week: $5 + 4 + 20 = 29 \text{ MOD } 7 = 29 - 28 = 1 =$ **Sunday**.

For October 16, 1962, we compute the year code as $62 + (62/4) \text{ MOD } 7 = 62 + 15 \text{ MOD } 7 = 77 \text{ MOD } 7 = 77 - 77 = 0$. Hence the day of the week for that date is $0 + 1 + 16 \text{ MOD } 7 = 17 \text{ MOD } 7 = 17 - 14 = 3 =$ **Tuesday**.

Note: When performing this calculation, ask for the year first, and compute the year code before asking for the month. If it takes a few seconds, that is okay. It gives the impression that you are trying to “visualize” the calendar for that year. Once you have computed the year code, the rest of the calculation goes very fast.

For years in the 2000’s, compute the year code for $2000 + X$ exactly as you would compute the year code for $1900 + X$, but subtract 1 from your final calculation. Thus, 2083 has year code $5 - 1 = 4$.

For years in the 1800’s add 2, and for years in the 2100’s (or 1700’s after the shift to the Gregorian calendar), add 4.

Thus 1883 has year code $5 + 2 \text{ MOD } 7 = 7 \text{ MOD } 7 = 0$;

1783 and 2183 have year code $5 + 4 \text{ MOD } 7 = 2$.

The calendar cycles every 400 years, so all years beginning 2200 or later can be reduced (MOD 400) to one of the years between 1800 and 2199.

Mathematical explanation: The basis for this algorithm is that most years have 365 days, which happens to be one more than a multiple of 7 (7 times 52 is 364). Thus, for most years, the calendar will shift by one day. However, on leap years (after February 29), the calendar shifts by an extra day. Beginning with the arbitrary choice of January having a month code of 1, and Sunday having the day code of 1, and that January 1, 1900 was a Monday, 1900 is given a year code of 0.

In 1983, the calendar has shifted $83 + 20 = 103$ times, that is, once for each year, and 20 more times for each leap year. (Incidentally, 1800, 1900, and 2100 are not leap years, but 2000 is a leap year.) And since shifting the calendar by 103 days is equivalent to shifting $103 - 98 = 5$ days, this is why the 1983 has a year code of 5. The month code reflects how much shifting takes place in each month. That is, since January has a month code of 1 and has 31 days, then February should have a month code of $32 \text{ MOD } 7 = 4$, and so on.

Practice these techniques, and you will soon find yourself the life of the mathematical party!

