

# A Generalized Vertex Truncation Scheme to Construct Intriguing Polyhedral Shapes

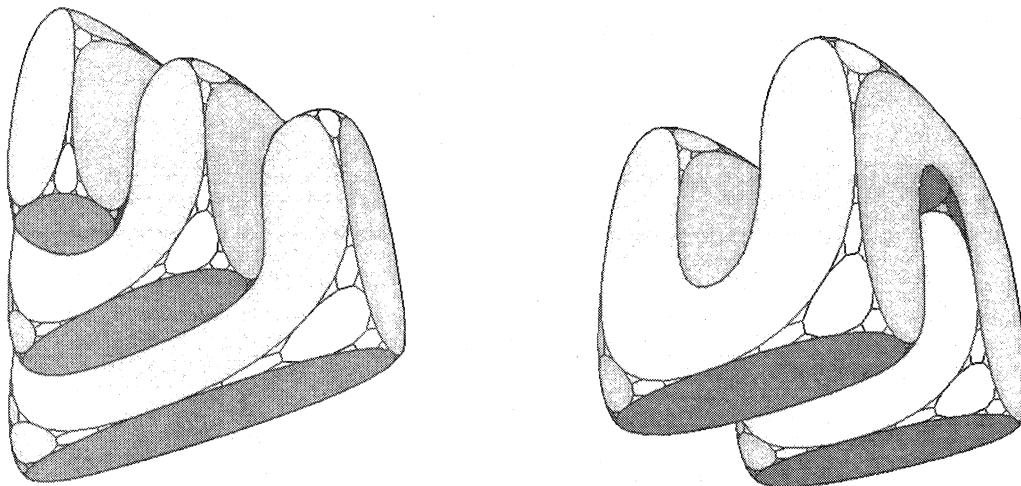
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## Abstract

In this paper, we present a method to create a new class of polyhedra. All the faces of these polyhedra are bounded by smooth (quadratic B-spline) curves and the face boundaries are  $C^1$  discontinuous everywhere. These polyhedral shapes are limit surfaces of a generalized vertex truncation subdivision scheme. We obtain an approximation of these smooth and fractal polyhedra by iteratively applying a new vertex truncation scheme to an initial manifold mesh. Our vertex truncation scheme is based on Chaikin's construction. If the initial manifold mesh is a polyhedra only with planar faces and 3-valent vertices, in each iteration we construct a polyhedral mesh in which all faces are planar and every vertex is 3-valent,

## 1. Introduction

One of the most exciting aspects of shape modeling and sculpting is the development of new algorithms and methods to create unusual, interesting and aesthetically pleasing shapes. Recent advances in computer graphics, shape modeling and mathematics help the imagination of contemporary mathematicians, artists and architects to design new and unusual 3D forms [11]. In this paper, we present such an unusual class of polyhedra that are contradictorily interesting, i.e. they are smooth but yet  $C^1$  discontinuous. To create these shapes we apply a polyhedral truncation algorithm based on Chaikin's scheme to any initial manifold mesh. Figure 1 shows two shapes that are created by using our approach.



**Figure 1:** *Two examples of smooth-fractal polyhedra that are created by our system..*

## 2. Background

One of the most notable artists of the last century, Escher, frequently applied mathematical concepts to create drawings of unusual 3D forms [6]. With the advance of computer graphics, many artists have begun to use mathematics as a tool to create revolutionary forms of artworks. There currently exist many contemporary artists such as George Hart [8], Helaman Ferguson [7], Bathsheba Grossman [9], Brent Collins and Carlo Séquin [20] who successfully combines art and mathematics to create unusual sculptures. These mathematical sculptors, who have a very noticeable presence in today's art scene, develop their own methods to model, prototype and fabricate an extraordinary variety of shapes.

Unusual shapes are especially interesting for architectural design. Interestingly shaped architectural structures are symbols of cities, regions, states, and even countries. In fact, recent advances in computer graphics and shape modeling help the imagination of contemporary architects to design new forms [11]. One most notable contemporary example of interestingly shaped buildings that have become symbols is Frank Gehry's Guggenheim Museum. The Museum, with its stunning titanium-clad forms, has been credited with vitalizing an entire city and region of Spain.

One of the major mathematical approaches to design unusual shapes is Fractal geometry. As told by Mandelbrot [12], Fractal geometry is often described as a "New Form of Art". Mandelbrot not only does not reject this idea, but he supports it by saying that "[with fractal geometry] all we do deal with is a new form of the controversial but ancient theme that all graphical representations of mathematical concepts are a form of art." Fractal geometry also introduced many new 3D forms such as Sierpinsky tetrahedron, Menger sponge, 3D subsets of 4D quaternion Julia and Mandelbrot sets. Shortly after their introduction by Mandelbrot, fractals have been widely used in contemporary architectural design. A large number of international architects such as Peter Eisenmann, Greg Lynn, Asymptote, Charles Correa, Coop Himmelblau, Carlos Ferrater, Arata Isozaki, Charles Jencks, Morphosis, and Eric Owen Moss have produced fractal rich contemporary architectural designs [1,10,14,15,16,17]. We can add some other designers like Karl Chu [22,13] who are concerned with genetic algorithms like L-systems or cellular automata. Architecture researchers also discovered that even before introduction of fractals, there exist many architectural examples that are rich in fractal aspects like facade environment relation in some vernacular dwellings [4], plan layouts and mass structures in the styles of modernistic architects such as Frank Lloyd Wright, Mies Van Der Rohe [15].

Our 3D forms are closely related one of the earliest known 2D fractal forms called Leibnitz packing of a circle (which themselves are a special case of an earlier form called Apollonian nets and gaskets) [12]. According to Mandelbrot, Leibnitz described it in a letter saying that **"Imagine a circle; inscribe within it three other circles congruent to each other and of maximum radius; proceed similarly within each of these circles and within each interval between them, and imagine that the process continues to infinity..."** Our operations convert each vertex of an initial polyhedron to a 3D form which looks similar to Leibnitz packing.

## 3. Methodology

Shape construction algorithms of fractal geometry are often given by a set of rules that are applied to an initial shape (see [3]). The problem with these shape construction algorithms is that they are hard to generalize. Each algorithm approaches its target shape regardless of the shape of the initial object. These algorithms do not allow construction of different target shapes from different initial shapes. Subdivision schemes provide a fresh alternative to fractal construction algorithms. They are conceptually similar to fractal constructions, i.e., they are also given by a set of rules that are applied to an initial shape.

However, the subdivision schemes have three advantages: (1) *their underlying rules (remeshing schemes) are mesh topological in nature*, (2) *the rules can simply be applied to any manifold polygonal mesh*, (3) *the limit shapes depend on initial shapes*. In this paper, we presents a subdivision scheme that can create unusually interesting shapes when it is successively applied to any manifold meshes, especially, to planar faced polyhedra in which all vertices 3-valent. Our subdivision scheme is a variant of polyhedral vertex truncation scheme.

### 3.1. Polyhedral Vertex Truncation.

Vertex truncation scheme is commonly used in Archimedean polyhedra construction [23]. Vertex truncation simply cuts each vertex with a planar surface. Mesh topological (remeshing) effect of vertex truncation is shown in Figure 2. As seen from this figure, after the application of a vertex truncation scheme, (1) *each polygon with n number of sides converts to a polygon with 2n number of sides*; (2) *each vertex with valence m becomes a polygon with m number of sides*; and (3) *all the new vertices become 3-valent*. In addition to these topological conditions, the polyhedral vertex truncation must satisfy one geometric condition: every initial and newly created polygonal face must be an equilateral and planar [23]. If the polyhedral vertex truncation is applied successively, because of the geometric condition each face eventually becomes a circle. And moreover the limit polyhedra closely resemble Leibnitz packing (in this case, the circles are not in the same plane).

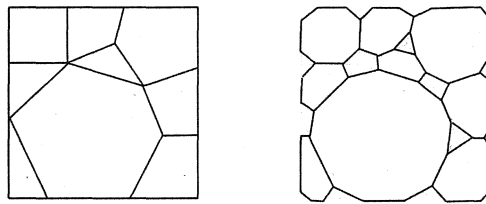


Figure 2: Remeshing effect of vertex truncation.

Although, the surface of the limit shape is not fractal, the edges of limit polyhedra show fractal property. Note that a shape is fractal if its Hausdorff-Besicovitch (or fractal) dimension is not an integer (This condition is sufficient but not necessary, i.e. a shape can be a fractal even when its fractal dimension is an integer)[12]. To compute fractal dimension, we need to find out how much the length of the edges is scaled, S, and how many more edges, N, are needed in each step. Then, the fractal dimension, D is computed as

$$D = \frac{\ln(N)}{\ln\left(\frac{1}{S}\right)}$$

when the number of the sides of largest polygon goes to  $\infty$ . Using the fact that in each iteration a regular n sided polygon goes to a regular 2n sided polygon, we can find that the length of the edges is scaled by

$$S = \frac{1}{2 \cos\left(\frac{2\pi}{n}\right)}$$

where n is the number of the sides of the largest polygon. Although, this number is not constant, note that when n goes to  $\infty$ , S approaches a constant

$$S_{n \rightarrow \infty} = \frac{1}{2}$$

Note that after the first iteration, all vertices become 3-valent and in each iteration the number of vertices increases by 3. In manifold surfaces in which every vertex is 3-valent, the following relationship exists between the number of vertices and edges:

$$3V_k = 2E_k \quad (1)$$

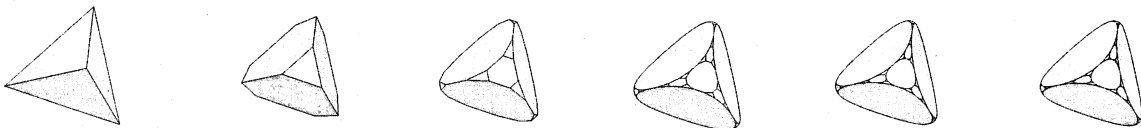
In this equation,  $V_k$  and  $E_k$  are the number of vertices and edges in iteration  $k$  respectively. Since, in each iteration the number of vertices increases by 2, the number of edges must also increase by 3 to satisfy the equation 1. In other word,  $N$  approaches 3. As a result, the dimension of polyhedral edges is found to be a non-integer

$$D = \frac{\ln(N)}{\ln\left(\frac{1}{S}\right)} = \frac{\ln(3)}{\ln(2)} = 1.5849;$$

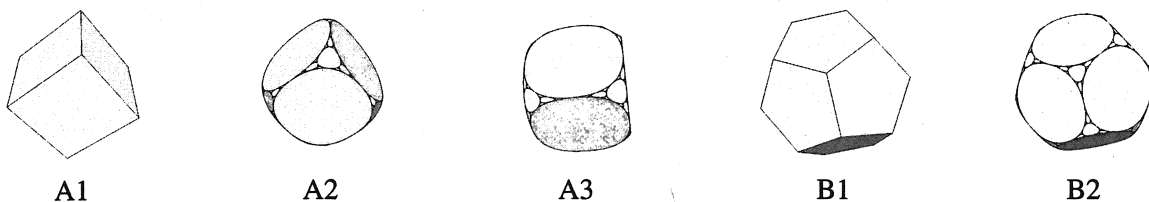
this number is the same as the dimension of Sierpinsky triangle.

### 3.2. New Vertex Truncation Scheme.

There are two problems using polyhedral vertex truncation to create smooth fractal polyhedra: (1) *Initial polyhedra that can be used for such an operation are extremely limited: Platonic polyhedra and some Archimedean polyhedra.* (2) *Polyhedral vertex truncation scheme is not stationary, i.e. it changes with each iteration.* To solve these problems we simply adapt Chaikin's algorithm [19,24] to vertex truncation. We use Chaikin's algorithm to determine the positions of the new vertices after a cut operation, i.e., we cut each line from  $1/4$  and  $3/4$ . Figure 3 shows five iterations of this scheme when it is applied to uniform tetrahedron. Figure 4 shows the effect of five iterations of the scheme when it is applied to uniform cube and dodecahedron respectively.



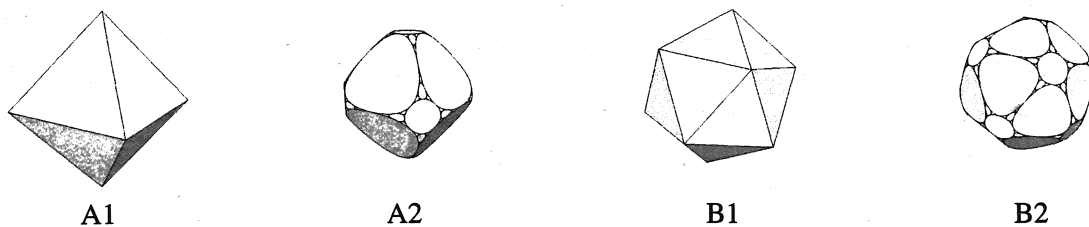
**Figure 3:** A tetrahedron and 5 successive vertex truncation operations applied to the tetrahedron.



**Figure 4:** A1, A2 and A3 are a cube and two views of its smooth fractal polyhedra that are created by 5 successive operation of vertex truncation. B1 is a dodecahedron and B2 is its smooth fractal counterpart created by 5 iterations of vertex truncation.

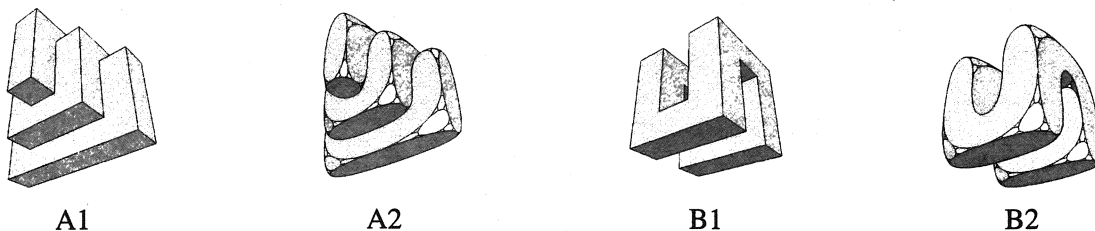
Unlike polyhedral vertex truncation, with this scheme, the faces, in the limit, do not go to circles; Chaikin algorithm approaches quadratic B-splines [19,24] and parametric polynomials cannot represent circles [2]. Similar to polyhedral vertex truncation, we can show that the dimension of the edges will also be  $D=1.5849$ . Comparing to polyhedral vertex truncation, this new vertex truncation scheme has several advantages. It is stationary and simple to implement. Moreover, the scheme is robust, i.e. it can be applied to any manifold mesh.

Although, the new scheme can be applied to any manifold mesh, if each face of initial mesh is planar and each vertex is 3-valent, then in each iteration we construct only planar faced polyhedral meshes, which can be very useful for Architectural applications. Initial 3-valent vertex condition guarantees that newly created faces after first application of the vertex truncation are planar faces. But, this is not necessary condition, i.e. for some special planar polyhedra such as regular octahedron and icosahedron as shown in Figure 5, the scheme still creates planar faces even with valence other-than-3 vertices. (Since all the vertices are 3-valent after the first iteration, each face created after second iteration is automatically planar.) To create interesting polyhedra with planar faces, the initial faces have to be planar, but they not have to be uniform. They can have any shape, e.g. stars and other non-convex shapes. The Figure 6 shows two such examples.

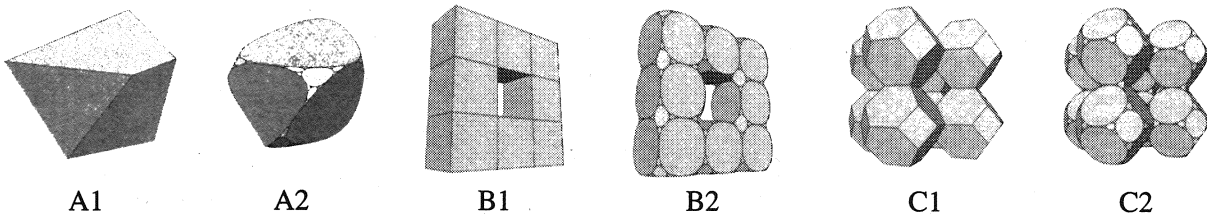


**Figure 5:** A1 is an octahedron and A2 is its 5 times vertex truncated version. B1 and B2 are an icosahedron and its 5 times vertex truncated counterpart. Notice that the faces are still planar despite the fact that 4-valent and 5-valence vertices of octahedra and icosahedra respectively.

If the initial meshes do not have only planar faces and 3-valent vertices, some faces of the constructed polygonal mesh may not be planar. The problem with such non-planar faces is that they may not be meaningfully reconstructed during rendering process. The problem can be theoretically worse during hardware rendering, but, as it can be seen in the screen-shot images shown in Figure 7, these non-planar faces with curved boundaries can be acceptably rendered (On the other hand, one of the bilinear quadrilateral faces of the deformed cube in Figure 7 do not look like a quadrilateral because of the triangulization in rendering stage). To build a real solid shape, the non-planar faces with curved boundaries need to be reconstructed by using a boundary interpolating subdivision scheme [24].



**Figure 6:** A1 is a non-convex polyhedron with non-convex planar faces and A2 is its 5 times vertex truncated counterpart. B1 is another non-convex polyhedron with non-convex planar faces and B2 is its 5 times vertex truncated version.

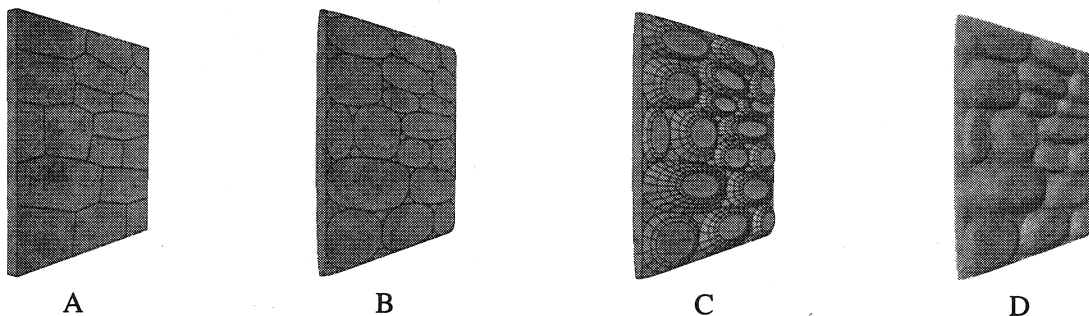


**Figure 7:** A1 and A2 are a deformed cube with non-planar faces and its 4 times vertex truncated version. B1 is a genus-1 shape with 4 valent-5 vertices that cause to construct non-planar faces in the first iteration and B2 is its 4 times vertex truncated version. C1 and C2 are a shape that is created by connecting eight space-packing polyhedra, truncated octahedron, and its 4 times vertex truncated version.

#### 4. Implementation

The algorithm above is implemented and included in our existing 2-manifold mesh modeling system, TopMod, [21] as an option. Our system is implemented in C++ and OpenGL, FLTK. All the examples in this paper were created using this system.

An unexpected usage of this approach is modeling walls that are made up stone as seen in Figure 8.D. To create such stone walls, we first subdivide a planar face into a set of polygons with valence-3 vertices (see Figure 8.A) and applied a few iteration of vertex cutting algorithm (see Figure 8.B). Then, using standard extrusion operation, we extrude each rounded face a few times with decreasing amount scaling and translation (see Figure 8.C).



**Figure 8:** A stone wall created by using our approach.

## 5. Conclusion

In this paper, we presented a new class of polyhedra. All the faces of these polyhedra are planes that are bounded by quadratic B-spline curves and the face boundaries are C1 discontinuous everywhere. These polyhedral shapes are limit surfaces of a simple subdivision algorithm that truncates the vertices. We obtain an approximation of these smooth fractal polyhedra by iteratively applying this new vertex truncation scheme to an initial manifold mesh. Our vertex truncation scheme is based on Chaikin's construction. If the initial manifold mesh is a polyhedron only with planar faces and 3-valent vertices, in each iteration we construct a polyhedral mesh in which all faces are planar and every vertex is 3-valent,

## 6. Acknowledgments

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