Cayley Tables as Quilt Designs

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1. Introduction

The algebraic structure of Cayley tables can be analyzed using color-coded tables. In our presentation we would like to discuss the use of colors to display characteristics of groups and monoids. The designs that are constructed by these color-coded tables provide the basis of several quilt designs.

2. Definitions

In order to talk about the structure of Cayley tables, we begin with some definitions.

2.1 Monoid. A set G together with a binary operation * is called a monoid if
(a) G is associative under * (For every a, b, and c in G, we have \(a * (b * c) = (a * b) * c\).)
(b) G has an identity element (There exists an element e such that for any a in G, \(a * e = e * a = a\).)

2.2 Group. A set G together with a binary operation * is called a group if
(a) G is a monoid
(b) Each element in G has an inverse (For each a in G, there exists an element b such that \(a * b = b * a = e\).)

2.3 Commutative Property. A binary operation * on a set G is said to be commutative if for every a and b in G, \(a * b = b * a\).

3. Colored Cayley Tables

The Cayley tables describing monoids have many beautiful symmetries and patterns embedded within them. Using colors to represent individual elements in the set, instead of more traditional symbols such as letters or numbers, allows one to easily recognize the properties of the groups or monoids. For example, from the Cayley table of \(\mathbb{Z}_6\) under multiplication (see Figure 1) one can see that the table describes a monoid that is commutative since the table has reflection symmetry about the main diagonal.
The reflection along the main diagonal is also apparent in the multiplication table of \( \mathbb{Z}_7 \) (see Figure 2).

In contrast to the commutative group of \( \mathbb{Z}_7 \) under multiplication (see Figure 2), the group \( S_3 \) is not commutative. Therefore, the Cayley table of \( S_3 \) in Figure 3 does not have reflection symmetry about the main diagonal.

Colored Cayley tables can help show the difference between a monoid and a group. For example, the multiplication table of \( \mathbb{Z}_7 \) shows that \( \mathbb{Z}_7 \) is a group (see Figure 2). To see this, notice that every column and every row has every color in the table. Similarly, \( S_3 \) is also a group (see Figure 3). In contrast, in the algebraic structure in Figure 1, some elements do not have an inverse.

In our presentation we plan to discuss Cayley tables of other algebraic structures. We will discuss how properties of the algebraic structures are reflected in the properties of these tables, including the visual differences between tables made with even, odd, prime, and composite numbers of elements, as well as cyclic versus noncyclic groups. Additionally, we plan to present several of these tables as quilts made from fabric.