Number Theory and Art

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Abstract

The Metallic Means Family (MMF), was introduced by the author [1], as a family of positive irrational quadratic numbers, with many mathematical properties that justify the appearance of its members in many different fields of knowledge, including Art. Its more conspicuous member is the Golden Mean. Other members of the MMF are the Silver Mean, the Bronze Mean, the Copper Mean, the Nickel Mean, etc.

Once upon a time I was walking through a thick forest when suddenly, I met a very old and small man, with a enormous long white beard. I was so surprised to find somebody in the loneliness that I asked him: Who are you? , hoping that he will be able to answer me in a language I could understand... Fortunately, my wish was accomplished and he told me this strange story:

Let me introduce myself: I am the Golden Mean

$$\phi = \frac{1+\sqrt{5}}{2} = 1{,}618...$$

the patriarch of the Metallic Means Family (MMF), so called because its members carry the name of a metal. I am very, very old and I am proud of having been and been yet a paradigm of beauty and harmony in human culture.

Kepler has said that I am "one of the treasures of geometry" [2] and I was considered by the mythical Plato [3] as the key to the physics of the cosmos. Indeed, I am present in the descriptive properties of the five Platonic polyhedra (tetrahedron, hexahedron or cube, octahedron, dodecahedron and icosahedron). But that is not all! Numbers like $\phi, 1/\phi, \sqrt{1+\phi^2}, 1/\phi^2, \sqrt{\phi}$ appear in many subtle ways related also to Archimedean semi-regular and Archimedean dual polyhedra.

A long, long time ago, I was present at the prehistorical temple of Stonehenge, built on Salisbury Plain, England, circa 3000-1000 B.C. For a long time, it was believed that this enormous monument was a Druid temple. But after long research, Gerald S. Hawkins and Alexander Thom [4] proved that it was an astronomical observatory, and could have been used to predict the summer and winter solstices, the vernal and autumnal equinoxes and eclipses of both the sun and the moon. Stonehenge consists of a sequence of concentric circles made of large monoliths, with an elliptical inner temple (known as the "Altar Stone"), whose axes are in a gold relation since its ratio is approximately 1,61: it is a golden ellipse! (Fig. 1).

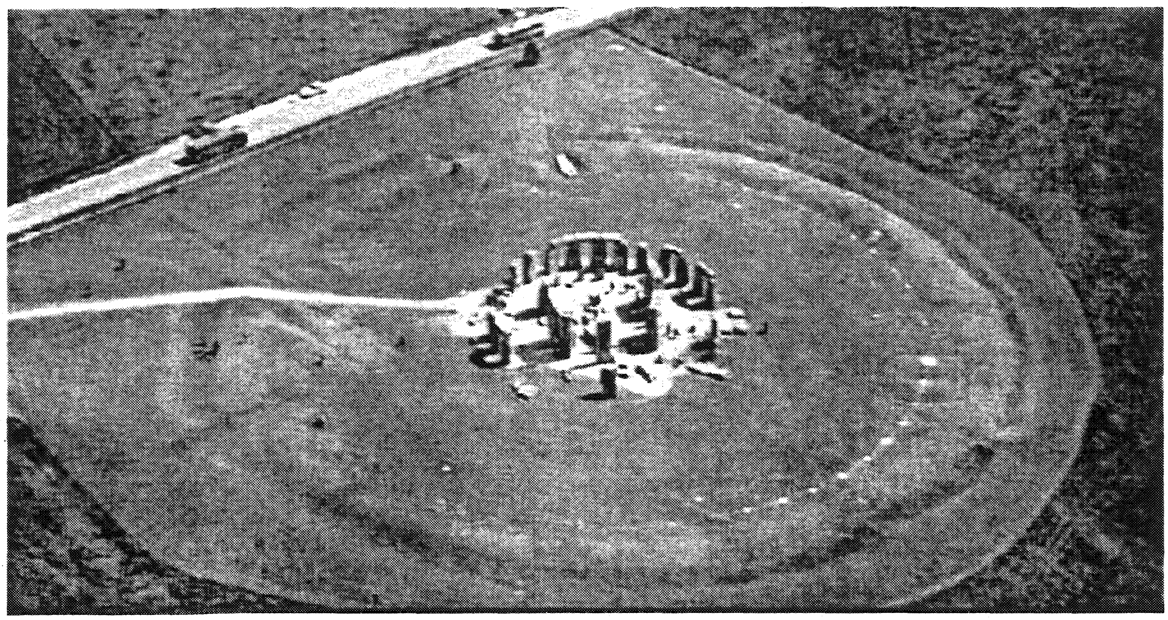

Fig. 1 - The temple of Stonehenge

I participated also at the building of the outstanding group of pyramids in Giza, Egypt: Mycerinus, Chefren and Cheops, circa 2500 B.C. Ask the famous traveler Herodotus how and why the Great Pyramid proportions were chosen to be $\phi, \sqrt{\phi}, \sqrt{1+\phi^2}$ (Fig. 2) and he will be stonelike in the face, as stonelike as the monuments perpetuating their millenary knowledge...

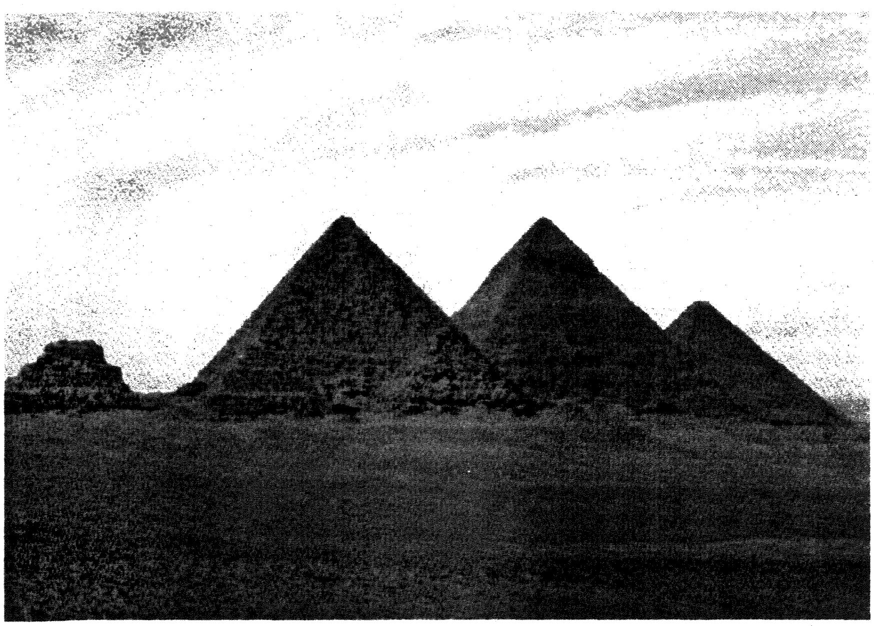

Fig. 2 - The pyramids of Mycerinus, Chefren and Cheops at Giza

And later on, I was at the Valley of the Kings, a burial site used by Egyptian rulers of the New Kingdom period (1570-1070 B.C.). It is located on the west bank of the Nile river, opposite

Luxor.There Rameses IV was buried at the Golden Chamber and the very chair of Tutankhamen was designed in golden proportions, as was noticed by Hambidge [5]. But these were not the only proofs of my influence: the attractive beauty of the well known Nefertiti statuette, created by sculptor Tutmes, who lived at the time of Pharaoh Amnhotep IV, is determined by a system of metric relations based on the Golden Mean [6].

Traveling to the West, you surely know the Parthenon, a white marble temple which is the greatest example of the classical stage of Greek art. It was built by architects Ictinos and Callicrates in 448-432 B.C. At the Doric front of it, I appear three times and if you take a look at the inner temple dedicated to Athene, you will notice that it is bounded by a golden rectangle of sides $\phi : 1$ (Fig. 3).

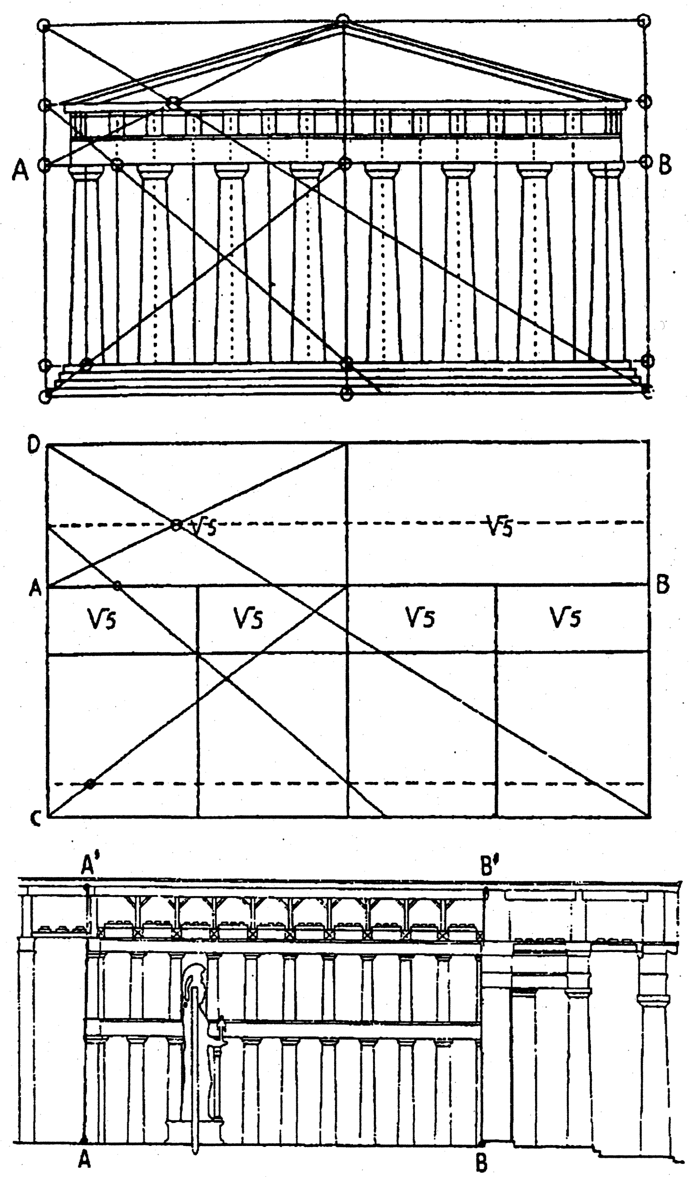

Fig. 3 - Three occurrences of the Golden Mean on the front of the Parthenon. Side elevation of the Parthenon showing the inner temle dedicated to Athene

Now, let us turn forth the clock a bit until we reach the Middle Ages. This period comprises approximately from the fall of the Western Roman Empire in 476 A.D. until the end of the Byzantine Roman Empire in 1453. Its more or less thousand years are usually divided into two well characterized periods: the High Middle Ages and the Low Middle Ages. The High Middle Ages extended from 476 until the fall of the Carolingian Empire and I consider that after the long dawn that has followed the gilt afternoon of classical culture, an unprecedented intellectual ferment developed and new cathedrals and monastic schools prospered. Gothic architects built some well known cathedrals during this period, e.g. Trondheim, Strasbourg, York, Notre Dame de Paris, Amiens, Bauvais, Cologne, etc. The most typical gothic cathedral designs include my symmetrical properties and geometrical consequences, as was proved in 1921 by the Norwegian archaeologist F. M. Lund [7] (see Fig. 4). He, who was in charge of the restoration of St. Olaf of Nidaros cathedral (today Trondheim cathedral) found that all the oblique directrices formed an angle of 63°26' with all horizontal as well as vertical lines. It is easy to verify that this angle is associated with my irrational part $\sqrt{5}/2$.

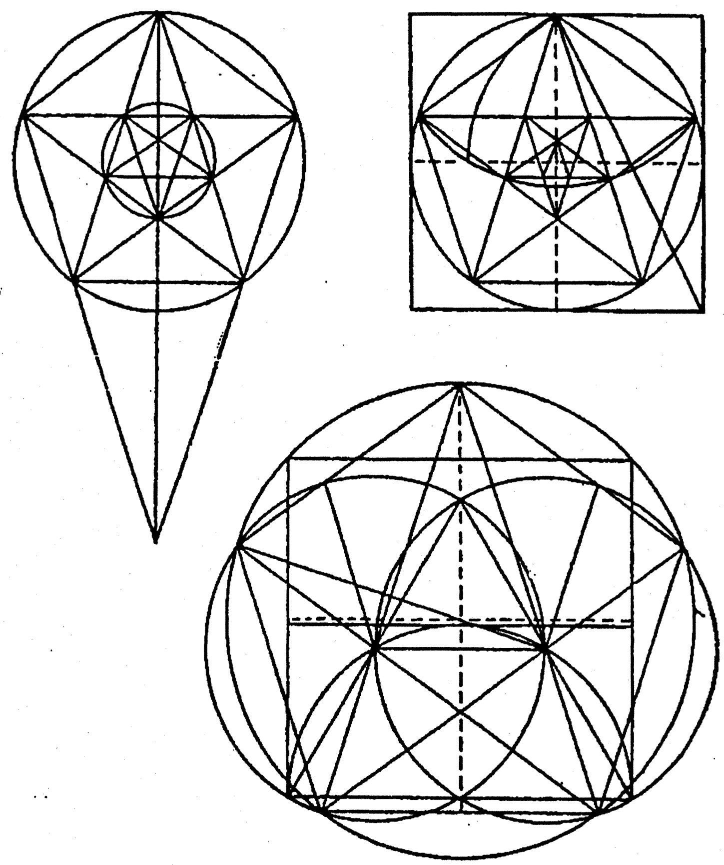

Fig. 4 - The most typical gothic cathedral designs

If you think I was present only at the construction of old monuments, you are wrong! I am an essential number in the description of plant symmetry properties and phyllotaxis, as well as in the analysis of spirals in sea animals and even in relations between different parts of human beings, as it was discovered by the German scientist Adolph Zeising circa 1850 [8].

Furthermore, in this century, I was taken as the basis of the proportion system called the Modulor, established by Le Corbusier [9]. The Modulor relates me to a human figure of 182 cm. (6 feet) tall and breaks up rectangular spaces according to the relationships between me and the human proportions.

I have also had a preponderant role in the system of musical proportions used in classical european music. In fact, assigning the "major chord" to the ratio 5/8 and the "minor chord" to the ratio 3/5, it is possible to generate the following sequences of major ratios

$$\frac{1}{1}, \frac{2}{3}, \frac{5}{8}, \frac{13}{21}, \frac{34}{55}, \ldots$$

that converges to $1/\phi$ and the one of minor ratios

$$\frac{0}{1}, \frac{1}{2}, \frac{3}{5}, \frac{8}{13}, \frac{21}{34}, \ldots$$

that also converges to $1/\phi$. The values of the sequence obtained by combining the two previous ones

$$\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \frac{21}{34}, \ldots$$

are the rational approximants of the continued fraction expansion of $1/\phi$:

$$\frac{1}{\phi} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}$$

Notice that my own continued fraction expansion

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{\ddots}} = [1,1,\ldots] = [\overline{1}]$$

is purely periodic and proves that <u>I am the most irrational of all irrationals</u>, because the sequence of rational approximants is the most slowly convergent of all the MMF!

Interesting to notice, all these integers are Fibonacci numbers, i.e. they form part of a Fibonacci sequence that is defined as a sequence of natural numbers determined by taking each number equal to the sum of the last two.

Now, let us talk about my family. I have a lot of cousins (really I do not know how many...) that also have a purely periodic continued fraction expansion, like the Silver Mean

$\sigma_{Ag} = [2,2,...] = [\overline{2}] = 1+\sqrt{2}$,the Bronze Mean $\sigma_{Br} = \dfrac{3+\sqrt{13}}{2} = [\overline{3}]$ and many others of the form $[\overline{n}]$.

Other relatives are the Copper Mean $\sigma_{Cu} = 2 = [2,\overline{0}]$, the Nickel Mean $\sigma_{Ni} = \dfrac{1+\sqrt{13}}{2} = [2,\overline{3}]$ and many periodic continued fraction expansions of the form $[\overline{m,n_1,n_2,...,n_n}]$.

Of course, they have <u>not</u> been so important as I was! But I have to admit that my first cousin, the Silver Mean, has also been used frequently as a proportion to compose music, build temples and palaces. For example, as the mathematician Jay Kappraff [10] noticed, the Roman system of architectonic proportions used precisely a numerical scheme based on the Silver Mean.

More than that, the "sacred cut" (so called by the geometer Tons Brunès [11]), reduces any length by a factor of $1/\sqrt{2}$ and four sacred cuts form the vertices of a regular octagon. This fact means that me and my first cousin are omnipresent in all groups of polyhedra as well as in Nature.

And what about the rest of my relatives? Even when we have in common many mathematical properties (see [1] for a detailed explanation of them), there are some differences in our behaviour which are fundamental in many modern applications.

In 1989, Gumbs and Ali [12], using Kohmoto's renormalization technique [13], analyzed different sorts of quasi-crystals and arrived to the following important conclusions:

1) The trace maps for the Metallic Means with purely periodic continued fraction expansions, are locally volume preserving.

2) If the Metallic Means have periodic continued fraction expansions, the trace maps are volume-non-preserving and the existence of a very large number of bounded orbits indicates the presence of "strange attractors", one of the main ingredients of a chaotic behaviour in a dynamical system.

The members of my family are intrinsically related with the onset from a periodic dynamics to a quasi-periodic dynamics and therefore, with the transition from order to chaos. In modern physics, my first cousin and I are related to the dimensions of the Cantorian space time of quantum physics, introduced by M. S. El Naschie [14]. These dimensions are connected with the KAM (Kolmogorov, Arnold and Moser [15][16][17]) theorem of Hamiltonian systems, the geometry of four manifolds and the 26-dimensional superstring spaces. By the way, an important consequence of KAM theorem is that orbits with an irrational winding number ratio are the most stable ones. Of course, as I am the most irrational number, I produce an orbit which is destroyed last when the energy of the system is increased!

Such a wide range of applications opens many roads to new inter-disciplinary investigations that will, undoubtedly, clear up the existent relations between Art and Technology, establishing a bridge among the rational scientific approach and the aesthetical emotion. And I hope this perspective could help you, human beings, to give to Technology, from which you depend increasingly for your survivorship, more sensible and sustainable features. This is my wish!

And all of a sudden, he flew away in the blue sky…

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