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An Introduction to the Tiling Properties of Precious Triangles

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Abstract

This introduction to the tiling properties of Precious Triangle covers the definition of Precious Triangles based upon the ratio $\pi \div 7$. It continues with examples of the unique way in which they can produce an infinite number of tiling patterns. It explains the iterative nature of the process as applied to Ad Hoc designs and tiling around a point. A discussion on the proportions of each tile is included which leads to some interesting conjectures. As a conclusion, some artistic examples of their usage is included

1. Introduction

This work explores the properties of Precious Triangles. In particular the creation of designs and patterns based on these triangles. The investigation were made as a result of following my hobby. In general the examples are the result of a program written in Vbasic.
Being a "lay" mathematician some of the descriptions will be less academic than normally found in a mathematical work. Nevertheless, I hope the examples will help in the understanding.

2. What are Precious Triangles?

The Precious Triangles described here are groups of triangles which can be used to form other sets of similar triangles using only the original set. This allows us to create an infinite series of designs using the original group of triangles. As a starting point I investigated triangles whose angles were multiples of $\pi \div 7$. In this introduction I will consider triangles whose angles are integral multiples of $\pi \div 7$. I have also investigated cases of $n = 3$ up to $n=10$. These are subjects of a separate article. I accept that there may well be examples of sets of triangles whose angles aren't multiples of $\pi \div 7$. It is, perhaps worth saying that the angle π is the mathematician's preferred measure of angles (in radians). This is exactly the same as 180 degrees. Most people know (or knew once!) that the sum of the interior angles of a triangle add up to 180 degrees. Another way of saying the same thing is that the interior angles of a triangle add up to π radians.

3. Precious Triangles based on $\pi/7$.

There are only four unique triangles whose angles are multiples of $\pi \div 7$. (for those who prefer degrees $\pi \div 7 = 180/7 = 25.714286$ degrees to six decimal places. This is another reason for using $\pi \div 7$.

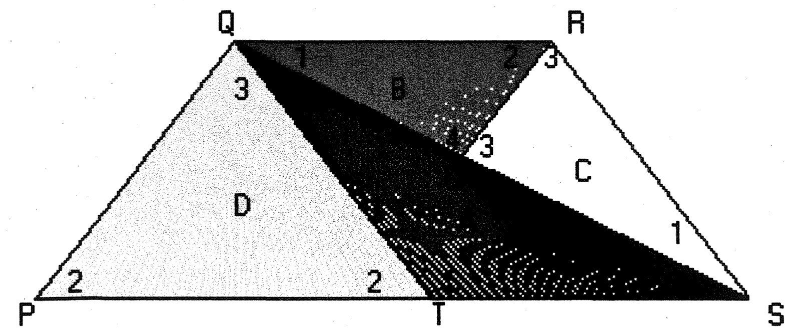

Figure 1: *four Precious Triangles forming a trapezoid*

I have called the triangles
A[1,1,5]
B[1,2,4]
C[1,3,3] and D[2,2,3].

A[1,1,5] means that for triangle A its angles have values $\pi \div 7, \pi \div 7$ and $5\pi \div 7$. Note the three numbers must add up to seven since the angles of a triangle add up to π radians. (or 180 degrees). Any other triangle using only multiples of $\pi \div 7$ will be similar to one of these four.

To qualify as Precious Triangles we must be able to create another set of four larger but similar triangles. In addition, the larger ones must have the same enlargement ratio.

Figure 2 illustrates how four similar triangles can be created from the original four triangles. It is easy to show that each large triangle is larger by a factor of $1+2\cos(2\pi/7)$ which has a value of 2.2469796, which just happens to be B(7)-1 where B(7) is the seventh Beraha number [1] I call this Ratio the Precious Ratio. As a matter of interest this is not the only regime for $\pi \div 7$. Each triangle in figure 2 is similar to the one to its left. Each triangle is larger by a factor of 2.246. I have coined the term Precious Ratio for this number. I have called this Ratio the Precious Ratio because I found that my similar work with triangles based upon $\pi \div 5$ resulted in the golden ratio [2] as the enlargement factor.

4. Geometry of the four triangles

With reference to figure 1, the trapezoid PQRS is constructed such that

$$PQ = QR = RS = d$$

The points T and U are such that

$$SU = ST = d$$

There are three distances to calculate UQ, TP and UR

$$QS = 2d\cos(\pi/7) = 1.8019377d$$

Hence $UQ = 0.8019377d$

And $TP = 2d\cos(2\pi/7) = 1.2469796d$

And $UR = 2d\cos(3\pi/7) = 0.4450419d$

Comparing the lengths of corresponding sides for the similar triangles above it is quite easy to show that each triangle has an enlargement factor of $1+2\cos(2\pi/7)$ which equals 2.2469796 [3].

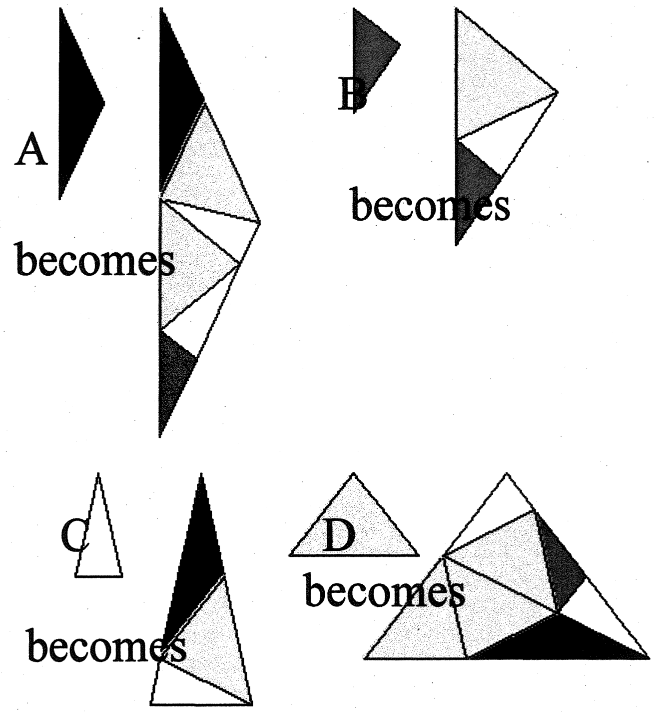

Figure 2: *Enlargement regime for the Precious Triangles*

5 Recursive Tiling Patterns

The trapezoid design PQRS can be constructed from the four triangular tiles A,B,C and D. As mentioned earlier they are the only four triangles whose angles are multiples of $\pi/7$. It is possible to construct a similar, larger trapezoid using the regime described in figure 2. It is important to note that the new design uses the same four tiles. This process could be repeated an infinite number of times. Each time a larger but similar design is produced, each being larger by a factor of 2.2469796 or the Precious Ratio. This process can be performed on any design, provided it is based on the original four triangles.

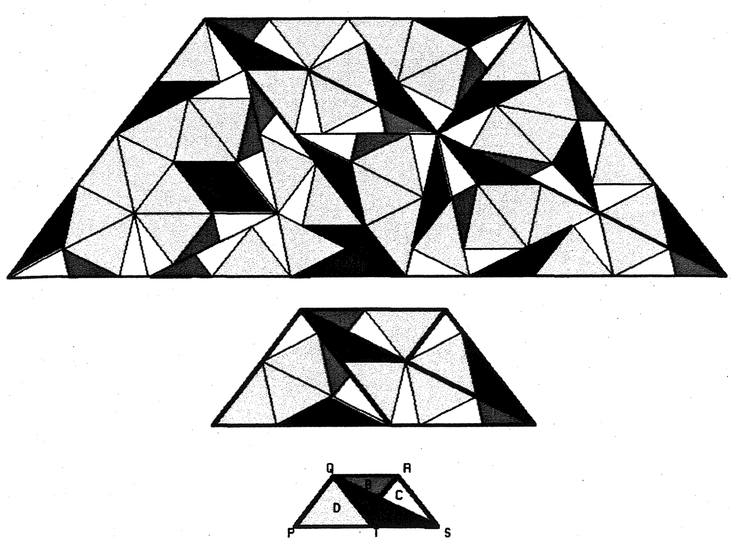

Figure 3: *Application of the tiling regime to the trapezoid.*

6 The Heptagon

If lines connecting the vertices of the heptagon are drawn then the triangles so formed happen to be similar to the Precious Triangles based on $\pi/7$. Unsurprisingly, therefore, the regular heptagon can be filled with the four Precious Triangles shown in figure 1. Figure 4 shows one way of achieving this.

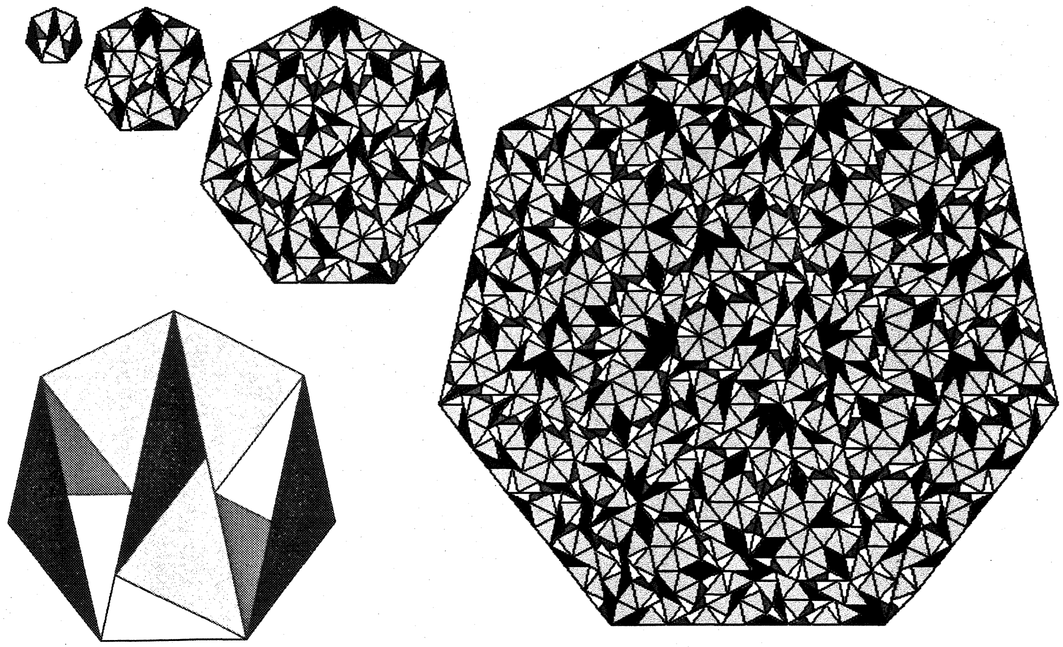

Figure 4: *Four generations of the heptagon formed from Precious Triangles*

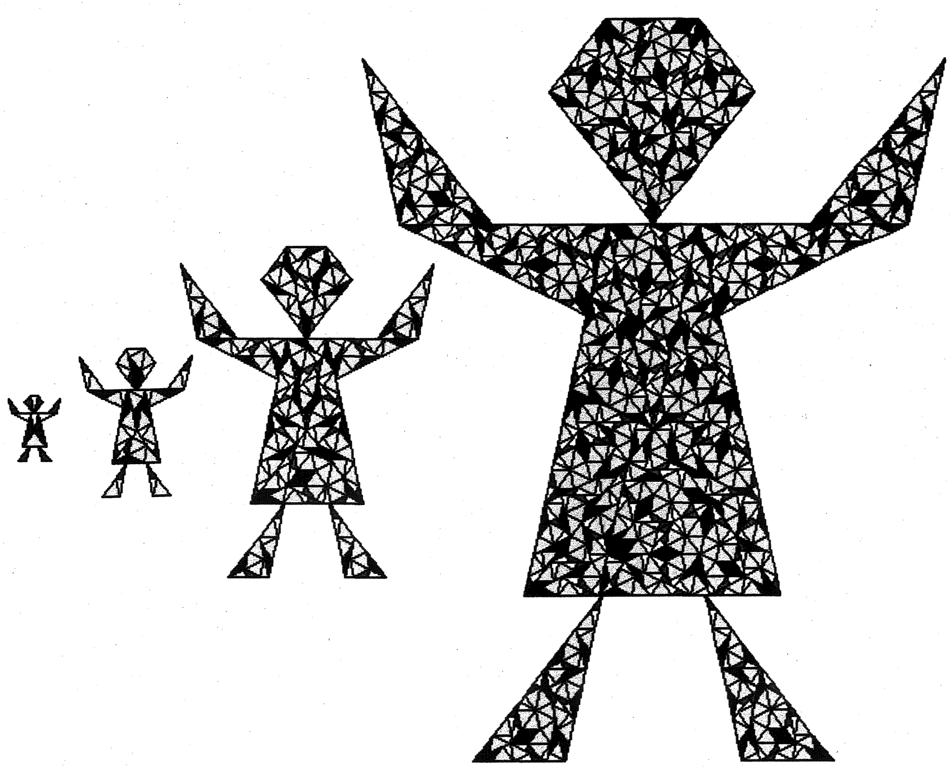

Figure 5: *An example of an Ad Hoc design called "The Alien"*

7. An example of an Ad Hoc Design

Figure 5 is an example of an Ad Hoc or improvised design: that is a design which has no particular structure or systematic constraint. This one I have called "The Alien". The smallest Alien, on the left is created from the four triangles. As we move to the right , each larger alien is created by repeated application of the regime. Each design uses only the original four triangles. If you look carefully it is possible to make out the lines of the original triangles.

8. Tiling around a Point

If we examine the four triangles, we notice that there are five different dimensions for the sides. Furthermore, the angles are multiples of $\pi/7$. This means that all the points of the triangles tessellating around a point lie on the corners of five concentric 14-ogons. Figure 6 illustrates the start of four designs. Completed designs are shown in figure 7. Figure 8 shows the design after a single iteration and figure 9 the result after 4 iterations.

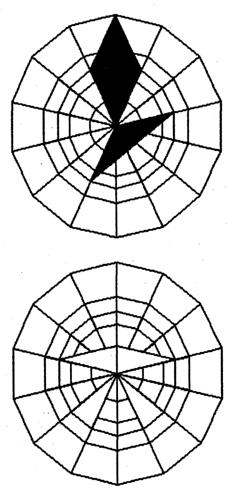

Figure 6: *Tiling around a point*

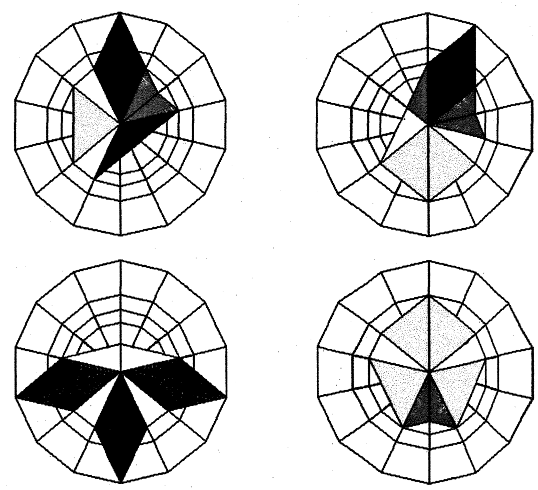

Figure 7: *completed examples of tiling around a point*

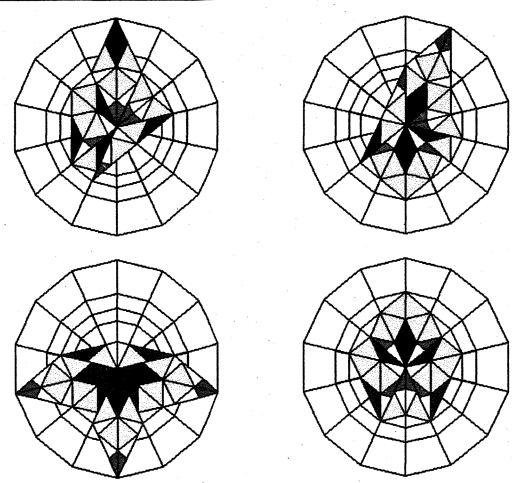

Figure 8: *the first generation tiling.*

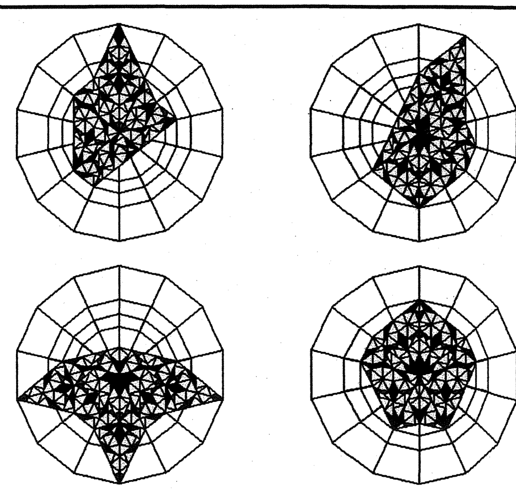

Figure 9: *the fourth generation tiling.*

9 Variations on a theme

The designs on this page are based on the designs in figure 7. In figure 10 the triangle D[2,2,3] has not been transformed. Gradually the design is tiled by more and more similar triangles, coloured light grey in this case. Clearly, other designs could be created by not transforming different triangles. Figures 11,12 and 13 show the result of not transforming the A, B and C triangles.

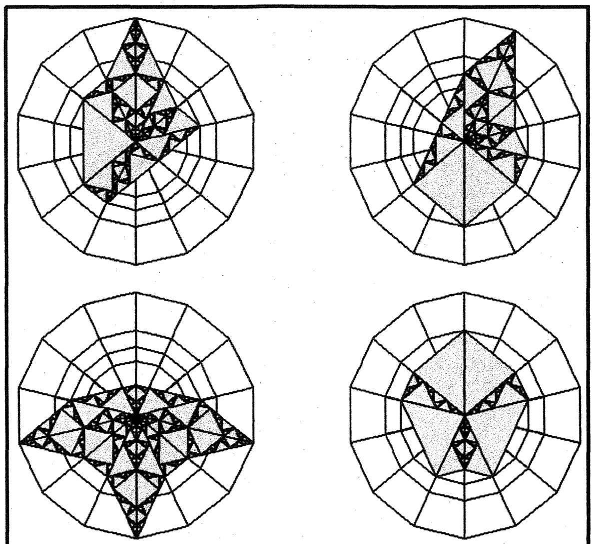

Figure 10: *These designs were created by not transforming the D[2,2,3] triangles.*

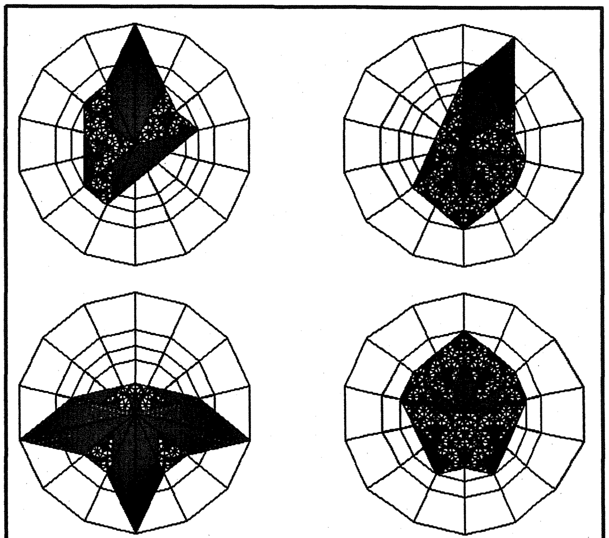

Figure 11: *These designs were created by not transforming the A[1,1,5] triangles.*

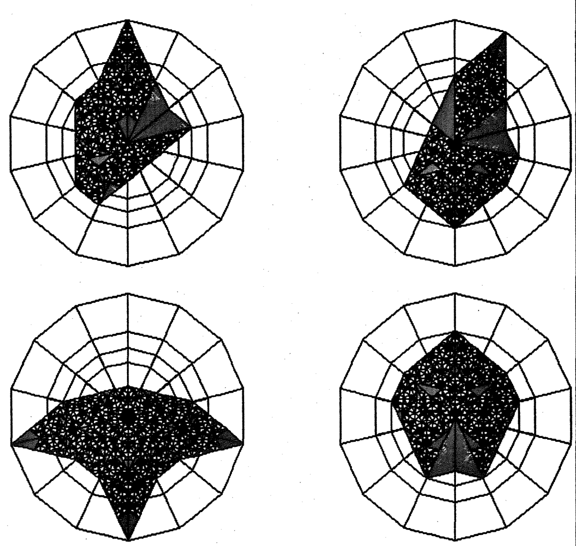

Figure 12: *These designs were created by not transforming the B[1,2,4] triangles.*

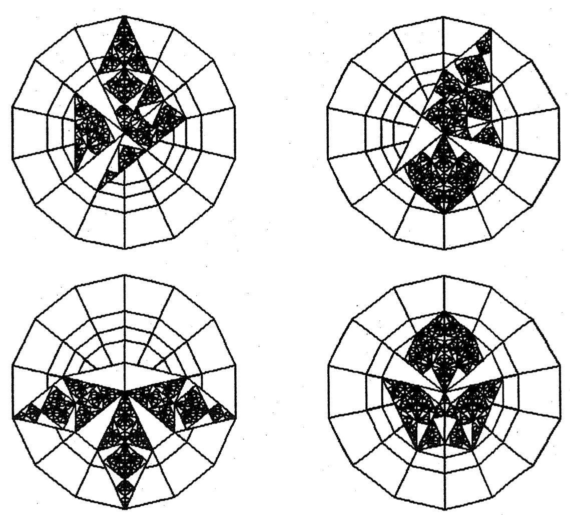

Figure 13: *These designs were created by not transforming the C[1,3,3] triangles.*

10 Some interesting Ratios.

The only ratio introduced so far is the Precious Ratio for the group of triangles. This ratio is the scale factor for the size of consecutive transformations and is denoted by Pr, where

$$Pr = 1 + 2\cos(2\pi/7) = 2.2469796 \text{ approximately}$$

This turns out to be B(7) - 1 where B(7) is the seventh Beraha constant.I have found that the Precious Ratio in cases other than $\pi/7$ also seem to be related to the Beraha Constant in a similar way. Why this is so will be the topic of a future investigation.
The results of a simulation [3] would suggest that the proportions for each triangle are

0.170915 for the A[1,1,5]
0.13706 for the B[1,2,4]
0.30797 for the C[1,3,3]
And 0.38404 for the D[2,2,3]

The thing that I found most interesting was that these proportions are independent of the numbers of each triangle in the original designs. The simulations for the first 8 recursions for the trapezoid, Alien, heptagon and for a single A[1,1,5] all resulted in the same proportion of each triangle in the final design.

Something that I found a little more amazing was that the ratio

A:Bn tends to Pr - 1

An:Dn tends to 1/Pr

and An:Cn tends to 1 - 1/Pr

Where Pr is the Precious Ratio for the group of triangles

The results of a simulation exercise confirm these conjectures to an accuracy of six decimal places.

I was somewhat more surprised when I simulated the ratios for the designs that kept the D[2,2,3] triangles and not transformed them. (see figure 10). Although it appears that the D triangles fill the design from an area point of view, it is far from true from a numbers point of view.
It appears from a simulation that the ratios of tiles, in this case, relate to the golden ratio rather than the Precious Ratio.
If we denote the golden ratio as Gr then the ratio

An:Bn tends towards Gr
Dn:An tends towards Gr^2
Dn:Bn tends towards Gr^3
Dn:Cn tends towards Gr
And An:Cn tends towards Gr - 1

The results of a simulation confirm the conjecture to an accuracy of six decimal places. Whilst I wasn't too surprised to find the golden ratio with $\pi/5$ triangles it was unexpected with $\pi/7$. I have been able to prove some of these results for $\pi/7$ and $\pi/5$. [3].

11 Est Werde Licht

The story of the creation in the first book of Genesis is, probably, the earliest scientific account ever recorded. The sequence of events contains a degree of logic and insight into the natural world, especially remarkable as it was first written thousands of years ago. The geometry consists of seven Precious Triangle designs, one for each day. The first depicts the creation of energy in the form of light. The second depicts the formation of water on earth, this being separate from the water in other galaxies. The third depicts the formation of dry land. The earth is embraced by the branches of a tree which supports the ecology of the world. The fourth depicts the sun, moon and stars. Their significance and magnitude were not appreciated when Genesis was written and hence may seem out of order from the viewpoint of modern science. Next the fish and birds are created in a sensible evolutionary order. The sixth depicts Adam, sitting on his haunches watching over the beasts in the fields. The seventh depicts Adam on the seventh day. Notice Adam's lines of symmetry, to denote the reflective nature of the Sabbath.

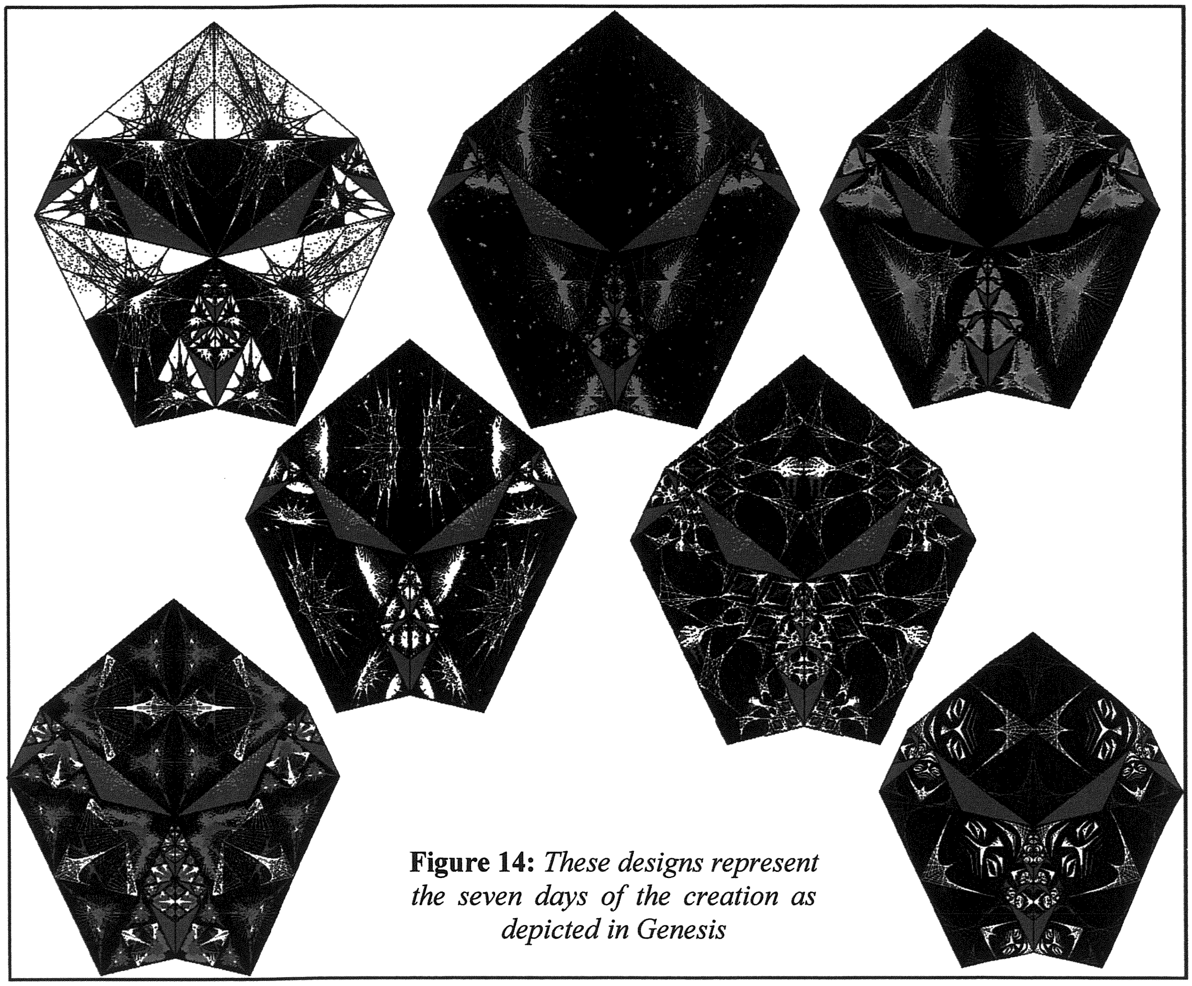

Figure 14: *These designs represent the seven days of the creation as depicted in Genesis*

13. References

[1] Weisstein, Eric W (Accessed 6.6.2002) *Beraha Constants* <http://mathworld.wolfram.com>

[2] Weisstein, Eric W (Accessed 6.6.2002) *Golden Ratios* <http://mathworld.wolfram.com>

[3] Spencer, Stanley J (2002) Tutorial *notes for ASAMA Conference*<http://pythagoras.org.uk>