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Classification and Phylogenetic Analysis of African Ternary Rhythm Timelines

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Abstract

A combinatorial classification and a phylogenetic analysis of the ten 12/8 time, seven-stroke *bell* rhythm timelines in African and Afro-American music are presented. New methods for rhythm classification are proposed based on measures of *rhythmic oddity* and *off-beatness*. These classifications reveal several new uniqueness properties of the *Bembé* bell pattern that may explain its widespread popularity. A new distance measure called the *swap*-distance is introduced to measure the non-similarity of two rhythms that have the same number of strokes (onsets). A swap in a sequence of notes and rests of equal duration is the location interchange of a note and a rest that are adjacent in the sequence. The swap distance between two rhythms is defined as the minimum number of swaps required to transform one rhythm to the other. A phylogenetic analysis using Splits Graphs with the swap distance shows that each of the ten bell patterns can be derived from one of two "canonical" patterns with at most four swap operations, or from one with at most five swap operations. Furthermore, the phylogenetic analysis suggests that for these ten bell patterns there are no "ancestral" rhythms not contained in this set.

1 Introduction

Consider the clock depicted in Figure 1, and assume the clock runs so fast that it makes a full revolution in about two seconds. Now set the clock ticking starting at "noon" (12 O'clock) and let it keep running for ever. Finally let it strike a bell on the hours of twelve, two, four, five, seven, nine, and eleven, for a total of seven strikes per clock cycle, with the first strike of the cycle at twelve. These times are marked with a bell in Figure 1. The resulting pattern rings out the predominant African rhythm time-line that has travelled to America and beyond, and has become the most well known of all the (12/8)-time bell patterns. It is known internationally mostly by its Cuban name, the *Bembé*, a name given to a Cuban feast celebrated with drums to entertain the *orishas* (divinities) [28]. In the following, simple mathematical arguments will be given that may explain why the *Bembé* has taken center stage among the 12/8 time bell patterns. Figure 2 shows five ways in which the *Bembé* bell pattern is usually notated. The fourth row depicts the time span of the rhythm divided into equal pulses consisting of the smallest convenient notes and rests. The final row shows the Bembé in the *Box Notation Method.* A commonly used convenient variant of box notation used in text documents is simply to use the letter "x" to denote the strike of the bell or note onset, and the period symbol "." to denote interval units between the note onsets. For example, the *Bembé* pattern of Figure 2 then becomes [x . x . x x . x . x . x]. In [35] a mathematical analysis of the six principal 4/4 time *clave* and bell patterns used in African and

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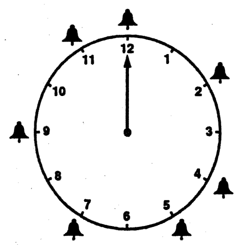

Figure 1: A clock that strikes a bell seven times in one cycle.

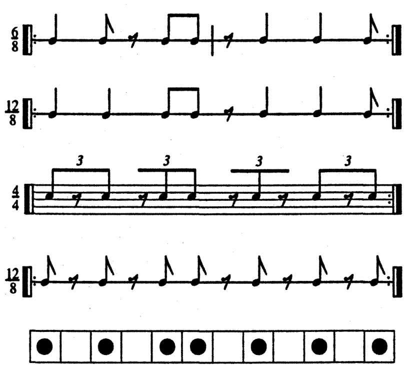

Figure 2: Five ways of representing the *Bembé* bell pattern.

Afro-American music was presented. Here we offer a similar study of the ten 12/8 time *ternary* African rhythm bell patterns (time-lines). The word *ternary* (also *triple meter*) is used to describe those rhythms that have the property that the number of pulses contained in the time span (in this case 12) is divisible by three.

There exist hundreds of timeline patterns for bells, claves and woodblocks traditionally used in music throughout Africa and America, and more recently in world music [1]. In this study however, we are concerned only with the seven-note 12/8 time rhythms. The total number of such possible rhythms is $12!/(7!)(5!) = 792$ if we do not make restrictions on how small or large the gaps between the notes may be. We can reduce this large number to an interesting small subset by restricting the maximum size of the inter-onset intervals. For this purpose a useful way to represent these timelines is as an *interval-vector* [29], i.e, a sequence of successive time intervals between the note onsets. Such a representation has been used for almost a century in the context of pitch-sets [22], [30] where it has also been called *successive-interval-arrays* [9]. The interval-vector for the *Bembé* is (2 2 1 2 2 2 1). In [35] a classification of rhythms was proposed based on permutations of the elements of the interval vectors. If one rhythm may be obtained from another by such a permutation then the two rhythms are said to belong to the same *interval combinatorial class*. One may ask how many interval permutations exist of the *Bembé* pattern (2 2 1 2 2 2 1). Note that these are *multisets* since repetitions of the elements are permitted [17]. We have seven objects (intervals) of two different types: two of class one and five of class two. Therefore the total number of different permutations of (2 2 1 2 2 2 1) is $(7!)/(2!)(5!) = 21$. In the following discussion a method for enumerating the twenty-one rhythms will be outlined. Although in this severely restricted class there are only twenty-one members, all the traditional music (that I am aware of) appears to use only ten of these, and none of the other 771 seven-note patterns. The ten commonly used rhythms

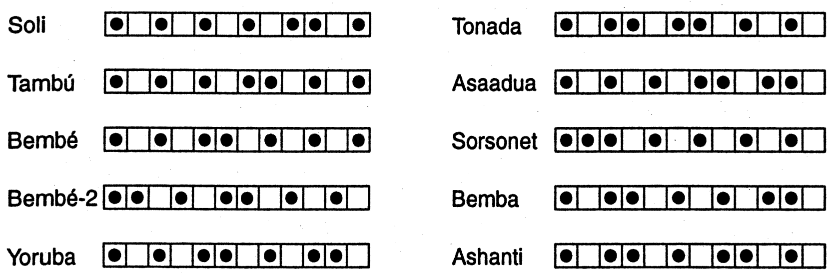

Figure 3: The ten 12/8 time bell patterns in box notation.

are known by many names in different countries, and there is as yet no international consensus on common terminology for all of them. For the purpose of this study I will call them: *Soli*, *Tambú*, *Bembé*, *Bembé-2*, *Yoruba*, *Tonada*, *Asaadua*, *Sorsonet*, *Bemba*, and *Ashanti*. Figure 3 depicts all ten rhythms in box notation.

A useful geometric representation for such cyclic rhythms is obtained by connecting consecutive note locations with edges to form a convex polygon. Such a representation not only enhances visualization for classification, but lends itself more readily to geometrical analysis. It has been used by Becker [5] to analyse Javanese Gamelan music, by McLachlan [24] to analyze rhythmic structures from Indonesia and Africa using group theory and Gestalt psychology, by London [23] to study meter representation in general, and most recently, it has been successfully used in a geometrical analysis of the six principal 4/4 time five-stroke clave and bell patterns used in African and Afro-American music [35]. The ten bell patterns of Figure 3 are represented as convex polygons in Figure 4, where a dashed line indicates the base of an isoceles triangle (two equal consecutive time intervals) and a solid diametral line indicates an axis of mirror symmetry. Note that, unlike the 4/4 time clave and bell patterns studied in [35], where the presence and number of an axis of mirror symmetry helped to distinguish between the different rhythms, here all ten bell patterns contain precisely one axis of symmetry. The 4/4 time and 12/8 time patterns also have very different interval combinatorial classifications. Whereas the six 4/4 time patterns fall into four different interval combinatorial classes [35], all ten 12/8 time patterns fall into one and the same combinatorial class. One may conclude that 12/8 time African rhythm bell patterns are more symmetrical and uniform than their 4/4 time counterparts. Another major difference between the 12/8 time and 4/4 time patterns is that, whereas the 4/4 time patterns are invariant to rotation, the 12/8 time patterns are not. A *k-ary necklace* is an equivalence class of k-ary strings under rotation. In this paper we are concerned with *binary* necklaces: the beads come in two colors, note onset interval and rest interval. A necklace is said to be of *fixed density* if the number of beads of one color is fixed [34]. Here we are concerned with binary necklaces of twelve beads with density seven. In the interval combinatorial class determined by the *Bembé* pattern there are only three distinct *necklace* patterns [20]. These three cannonical *necklaces* are shown in Figure 5 with their axes of mirror symmetry in the vertical position and with the two short intervals in the upper semi circle. Cannonical pattern number I generates the *Sorsonet* pattern. Cannonical pattern number II generates the *Soli*, *Tonada*, and *Asaadua* patterns. Cannonical pattern number III generates the *Bembé*, *Bembé-2*, *Tambú*, *Yoruba*, *Bemba*, and *Ashanti* patterns. The numbers I, II, and III assigned to the cannonical configurations correspond to the minimum distance (in terms of the number of intervals) that separates the two short intervals. It is clear that bell patterns with a larger separation (longer sequence of rests) between the two short intervals are prefered. The cannonical pattern number III has the largest possible separation between the two short intervals.

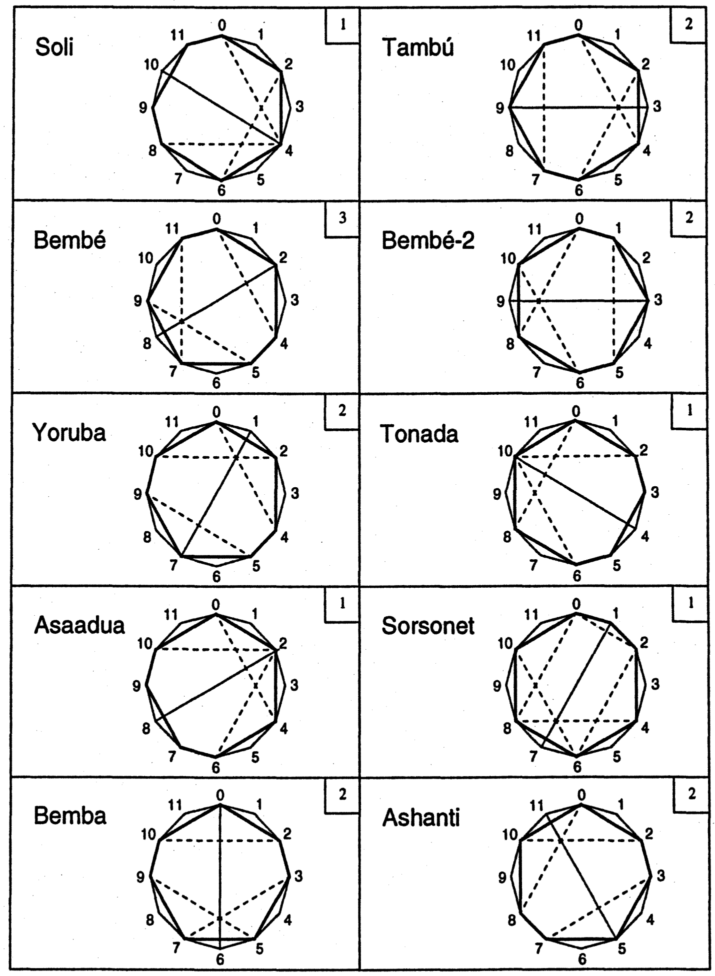

Figure 4: The ten 12/8 time bell patterns represented as convex polygons.

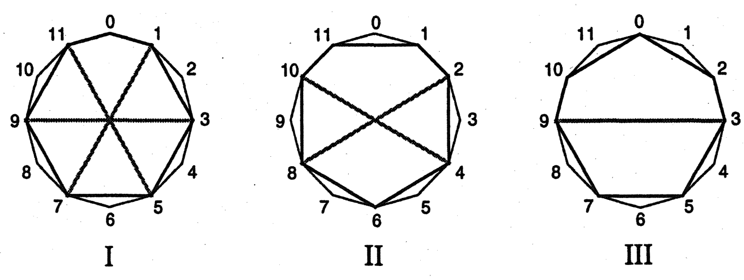

Figure 5: The three cannonical *necklace* patterns that generate all the rhythms, and the number of equal bi-partitions in each.

2 The Ten 12/8 Time Bell Patterns

The *Rhythm Catalog* of Larry Morris on the World Wide Web [25] contains a rhythm called *Soli* from West Africa with the bell pattern given by [x . x . x . x . x x . x]. The *Soli* actually has three bell patterns including the seven-stroke pattern included in this study, which is the pattern most frequently used in the *Soli* rhythm [15].

The *Tambú*, given by [x . x . x . x x . x . x], is found in several places in the Caribbean, including Curaçao, where it goes by this name [33]. This bell pattern is also common in West Africa and Haiti [29]. In Haiti it is used for the *Yanvalou*, *Zepaule*, *Camberto*, and *Mahi* rhythms [25]. This rhythm is sometimes called the *long* African bell pattern [13].

The rhythm denoted by [x . x . x x . x . x . x], is probably the most (internationally) well known of all the African ternary timelines. Indeed, the master drummer Desmond K. Tai has called it the Standard Pattern [19]. In West Africa it is found under various names among the Ewe and Yoruba peoples [29]. In Ghana it is played in the *Agbekor* dance rhythm found along the southern coast of Ghana [8], as well as in the *Bintin* rhythm [25]. It is also the bell pattern of the *Agbadza* rhythm for a recreational dance of the *Ewe* people of eastern Ghana and Togo (see chapter 22 of Collins [11]). It is played in the *Zebola* rhythm of the *Mongo* people of Congo, and in the *Tiriba* and *Liberté* rhythms of Guinea [15]. This bell pattern is equally widespread in America. In Cuba it is the principal bell pattern played on the *guataca* or hoe blade in the *Bata* rhythms. For example, it is used in the *Columbia de La Habana*, the *Bembé*, the *Chango*, the *Eleggua*, the *Imbaloke*, and the *Palo*. The pattern is also used in the *Guiro*, a Cuban folkloric rhythm [21]. In Haiti it is called the *Ibo* [25]. In Brazil it goes by the name of *Behavento* [25]. This rhythm is sometimes called the *short* African bell pattern [13]. In this study I shall refer to it by its most popular international name: the *Bembé*.

In Cuba sometimes the *Bembé* rhythm contains two bell patterns; the pattern described in the preceeding, played on the *guataca* or hoe-blade, and the pattern [x x . x . x x . x . x .], played on a low pitch bell [25]. For this reason this secondary bell pattern will be referred to here as *Bembé-2*. This pattern is also a hand-clapping pattern used in Ghana [29].

The bell pattern [x . x . x x . x . x x .] is widely used in sacred music among the Yoruba people of West Africa [29]. It is also used in Cuba with the *Columbia* rhythm [32]. The name *Yoruba* will be used for this bell pattern.

The *Tonada* in Cuba is a type of song that illustrates clearly the fusion between the singing style of Andalucia in Spain, and the African rhythms of Cuba. The bell pattern used in the *Tonada* is [x . x x . x x . x . x .] [32]. In the Caribbean it also appears in Martinique where it is used in the *Bélé* (bel-air) rhythm that accompanies a music and dance originating in the time of slavery [14]. In Africa this pattern is used in Ghana by the Ashanti people [29] and by the *Akan* people in *Adowa* music [26], [27]. It is also used in the *Mandiani* rhythm of Guinea [15].

The *Asaadua*, expressed as [x . x . x . x x . x x .], is used in processional music of the *Akan* people of central western Ghana [16]. It is played on a *dawuro* bell (also called *atoke* in other parts), a hollow boat-shaped iron bell with a piercing hi-pitched tone that cuts through a score of loud drums. It is also used in the *Kakilambe* and *Sokou* rhythms of Guinea [15].

The *Sorsonet* bell pattern given by [x x x . x . x . x . x .] is used by the *Baga* people of Guinea [15] and does not appear to be widely used.

The bell pattern denoted by [x . x x . x . x . x x .] is played with a rhythm found in Northern Zimbabwe called the *Bemba* [29] (not to be confused with the *Bembé* from Cuba).

The pattern [x . x x . x . x x . x .] is used by the *Ashanti* people of Ghana in several rhythms [29].

It is used in the *Dunumba* rhythm of Guinea [15], and by the *Akan* people of Ghana [26] as a juvenile song rhythm. It is also a pattern used by the *Bemba* people of Northern Zimbabwe, where it is either a hand-clapping pattern, or played by chinking pairs of axe-blades together [19].

3 Why is the Bembé Bell Pattern so Special?

3.1 Isomorphisms with Musical Scales

The reader familiar with the piano and the *diatonic* scale may have noticed the similarity between this pitch pattern and the time pattern of the *Bembé*. If we associate the notes and rests of the *Bembé* time pattern with the white and black keys, respectively, of the diatonic scale pitch pattern (for the *even tempered* piano) then we obtain an exact isomorphism between the two. Remarkably, Pressing [29] has discovered exact isomorphisms between almost all rhythm timelines and pitch patterns (scales) found in world music (see also Rahn [31]). Since the seven white keys of the diatonic scale (C, D, E, F, G, A, B) are so fundamental and important in Western music theory, it is not surprising that the study of this pattern has received a great deal of attention. Several mathematical properties have been suggested as testimony to the specialness of this pattern.

For example, the diatonic scale is generated by the so-called *circle-of-fifths* [6] in which we mark every fifth tone until seven tones are marked and then select all marked tones. If we apply this method instead in the rhythmic domain to generate seven onsets we obtain the canonical necklace pattern III of Figure 5 of which the *Bembé* is a member.

Duncan [12] describes a percolation algorithm that starts with the pattern [x x x x x x x . . . . .] and moves the x's to the right with the goal of producing a less dense pattern. When the algorithm stops the final pattern obtained is the diatonic scale in the pitch domain or the *Bembé* pattern in the time domain. Several group-theoretic [4] and combinatoric [20] properties of the diatonic scale have also been discovered. In the following subsections we add several additional simple characterizing properties of the *Bembé* pattern to this growing list.

Clough and Duthett [10] defined the notion of *maximally even sets* with respect to scales represented on a circle. Block and Douthet went further by proposing a measure of *evenness* [6]. Their measure simply adds all the circle-chord lengths determined by pairs of notes in the scale. These definitions may be readily applied to rhythms. It turns out that within the set of rhythms consisting of seven onsets in a bar of twelve units, the *Bembé* pattern, which corresponds to the diatonic scale in Figure 5 III, is a maximally even set. Interestingly, the two configurations that rank just below the *Bembé* pattern using the Block-Douthett evenness measure, are the ascending melodic minor scale, corresponding to Figure 5 II, and the whole-tone plus one scale, corresponding to Figure 5 I. Therefore the cannonical necklace patterns in Figure 5 correspond to the three most even rhythms.

3.2 Measuring Rhythmic Oddity

Simha Arom [3] defines a rhythm as having the *rhythmic oddity* property if no two onsets partition the entire interval into two subintervals (bi-partition) of equal length. We have also seen that none of the ten seven-stroke bell patterns has this property. However, we may define a measure of the amount of rhythmic oddity of a rhythm by the *number* of bi-partitions of equal length that it admits. The fewer equal bi-partitions a rhythm admits, the more rhythmic oddity it posesses. Figure 5 shows the three necklace patterns with the number of equal bi-partitions contained in each. Thus the *Bembé* bell pattern belongs to the class of rhythms that have maximum rhythmic oddity.

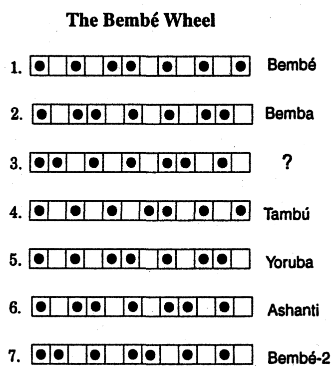

Figure 6: The Bembé wheel and the six known rhythms it generates.

3.3 Measuring Off-Beatness

In this section a measure of the *off-beatness* of a rhythm is introduced. A twelve-unit interval may be evenly divided (with no remainders) by *four* numbers greater than one and less than twelve. These are the numbers six, four, three and two. Dividing the twelve unit circle by these numbers yields a bi-angle, triangle, square, and hexagon, respectively. African music usually contains some drum or other percussion instrument that will play at least one or a portion of these patterns. In polyrhythmic music these four patterns form the possible underlying even pulses. Two of the patterns (bi-angle and square) are binary pulses and two (triangle and hexagon) ternary pulses. Therefore notes played on other positions are off-beat in a strong sense. There are four positions not used by these four even pulse patterns. They are positions 1, 5, 7, and 11. A rhythm that contains an onset in at least one of these four positions will be said to contain the *off-beat* property. A measure of the *off-beatness* of a rhythm is therefore the number of onsets it contains in these four positions. This number is indicated in the upper right-hand corner of each box in Figure 4. The highest value of *off-beatness* is three and only the *Bembé* realizes this value.

4 Rhythm Wheels

Some authors and teachers have noticed that if the *Bembé* bell pattern is played by starting on the fourth onset one obtains the *Tambú* bell pattern [13]. Others have gone further by suggesting pedagogical exercises in which one practices by starting the *Bembé* with each of its seven onsets acting as the first downbeat. In fact, in unpublished work, Gary Harding of Seattle has given this rhythm generation method the name *Wheel.* Several Internet sites are devoted to the *Bembé* Wheel which is shown in Figure 6. However, it has not been realized in the popular literature that this simple technique actually generates other traditional rhythms. The second bell pattern in Figure 6 is obtained by starting the *Bembé* on the second onset: the *Bemba* pattern. The third pattern is obtained by starting the *Bembé* on the third onset, and so on. As Figure 6 shows, this generation method yields seven patterns of which six are used in traditional African music. As for the third pattern [x x . x . x . x x . x .], I have not been able to find it anywhere. Note that the seven bell patterns of the *Bembé* Wheel are obtained by appropriate rotations of the cannonical necklace pattern number III in Figure 5. Clearly, just as cannonical necklace pattern number III can be made into a wheel, one can make a wheel out of the two other cannonical patterns. The wheel generated from the cannonical necklace pattern number II will be called the *Tonada* wheel and is

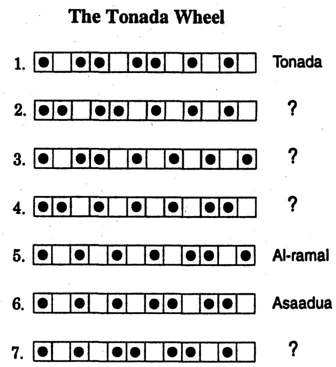

Figure 7: The *Tonada* wheel and the three known rhythms it generates.

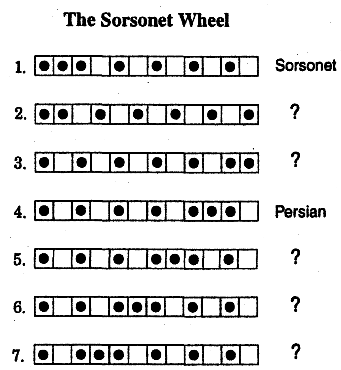

Figure 8: The *Sorsonet* wheel and the two known rhythms it generates.

depicted in Figure 7. The *Tonada* bell pattern generates only two other patterns that I have been able to find. One is the *Asaadua* (number six) from West Africa and the other (number five) is the *Al-ramal* from Arab and Persian music. This latter pattern dates back to books on rhythm written by SafI-al-DIn in the thirteenth century [36]. Interestingly, SafI-al-DIn depicted the rhythms as a circular "pie" chart divided into equal slices of pie. Each slice corresponded to a time unit and the slices corresponding to the onsets of the notes were shaded black. The wheel generated from cannonical necklace pattern number I will be called the *Sorsonet* wheel and is depicted in Figure 8. The *Sorsonet* wheel generates only one other pattern that I encountered in the literature. This is pattern number four, a Persian rhythm called *kitaab al-adwaar* that also dates back to SafI-al-DIn [36]. Comparing the three wheels *Sorsonet*, *Tonada*, and *Bembé* corresponding to the three cannonical necklace patterns, one notices that only two patterns are used from necklace I, three from necklace II, and six from necklace III. This suggests a clear direction of preference towards the patterns that have the two short intervals situated as far from each other as possible (or as evenly dispersed as possible) within the cycle.

The reader may have noticed that each of the three wheels generates exactly seven rhythms with this rotation method, for a total of twenty-one, which coincides with the number of different permutations of the interval vector (2212221). This is no coincidence. Indeed, all the permutations may be enumerated in this way by first generating the necklaces with a variety of existing algorithms [34] and then generating the wheel for each necklace.

Swap Distance Matrix

	Soli	Tambú	Bembé	Bembé-2	Yoruba	Tonada	Asaadua	Sorsonet	Bemba	Ashanti
Soli		1	2	7	3	6	2	9	4	5
Tambú			1	6	2	5	1	8	3	4
Bembé				5	1	4	2	7	2	3
Bembé-2					4	1	5	2	3	2
Yoruba						3	1	6	1	2
Tonada							4	3	2	1
Asaadua								7	2	3
Sorsonet									5	4
Bemba										1
Ashanti										
Σ	39	31	32	35	23	29	27	51	23	25

Figure 9: The swap distance matrix of the ten rhythms.

5 Phylogenetic Analysis of Rhythms

5.1 The Swap Distance

At the heart of any algorithm for comparing, recognizing or classifying a rhythm lies a measure of the similarity between two rhythms. There exists a wide variety of methods for measuring the similarity of two rhythms represented by a string of symbols. When the two strings are binary sequences a natural measure of distance or non-similarity between them is the Hamming distance: the number of places in the strings where elements do not match. Other approaches use the Euclidean distance between the interval vector representations of rhythms. These methods are more appropriate than the Hamming distance. However, for the phylogenetic analysis of rhythms, the *swap distance* proposed in the following is more natural.

A completely different approach to measuring the dissimilarity between two strings computes the amount of "work" required to transform one string into the other. Such an approach is common in bioinformatics where the two strings to be compared are chain polymers and the work is measured by the minimum number of basic operations required to transform one molecule into the other. The type of basic operation used varies and usually models some kind of mutation relevant to evolution (see [2] and the references therein). The problem of comparing two binary strings of the same length with the same number of one's suggests an extremely simple operation that will be called a *swap*. A swap is an interchange of a one and a zero (onset interval and rest interval) that are adjacent in the string. The swap distance between two rhythms is the minimum number of swaps required to convert one rhythm to the other. For example the Bembé rhythm [x . x . x x . x . x . x] can be converted to the *Tonada* rhythm [x . x x . x x . x . x .] by a minumum of four swaps, namely interchanging the third, fifth, sixth, and seventh strokes with the corresponding rests preceeding them. Such a measure of dissimilarity appears to be more appropriate than the Hamming distance between the binary vectors or the Euclidean distance between the interval vectors, in the context of rhythm similarity. The distance matrix for the ten seven-stroke bell patterns is shown in Figure 9 where the bottom row indicates, for each rhythm, the sum of the swap distances to all the other rhythms. We see that both the *Yoruba* and the *Bemba* are matched for low scores (23), indicating

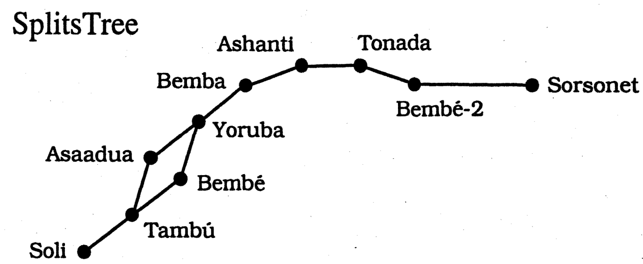

Figure 10: The SplitsTree constructed for the distance matrix in Figure 9.

that these two rhythms are more similar to all the rhythms than any other. At the other extreme lies the *Sorsonet* with a score of 51, making it the maverick in the group. The phylogenetic tree computed from this distance matrix is a more revealing structure, as we shall see in the following.

5.2 Phylogenetic Analysis

As was previously demonstrated with the 4/4 time *clave* and bell patterns studied in [35], phylogenetic trees provide a useful tool for visualizing the interrelationships between the rhythms as well as determining economical mechanisms for their generation. These mechanisms may in turn shed light on the evolution of such rhythms. In [35] the distance used was the Euclidean distance between the interval vectors, and the phylogenetic analysis used classical phylogenetic trees. One weakness of classical phylogenetic trees is that they impose a *tree* structure on the data even if the underlying structure is not a tree. However, one may be interested in knowing how appropriate such a tree structure is. In the bioinformatics literature there exist new techniques which provide this information in a graph that is a generalization of a tree. One notable example is the *SplitsTree* [18]. Like the more traditional phylogenetic trees, the Splits Graph is a drawing in the plane with the property that the distance in the drawing between any two nodes reflects as closely as possible the true distance between the corresponding two rhythms in the distance matrix. However, if the tree structure does not match the data perfectly then edges are split to form parallelograms whose size is proportional to the missmatch. Thus the *SplitsTree* may in fact be a graph that is not a tree and has cycles. The *SplitsTree* constructed from the distance matrix of Figure 9 shown in Figure 10 bears this out. The structure is almost a chain except for the four-cycle determined by the *Bembé*, *Yoruba*, *Asaadua*, and *Tambú* rhythms. From the *SplitsTree* several additional properties are immediately evident. Only the *Sorsonet* does not have swap distance one to any other rhythm. The diameter of the graph (two most distinct rhythms) is determined by the *Sorsonet* and the *Soli*. The *center* of the graph (i.e., the vertex that minimizes the maximum distance to any other vertex in the graph) is determined by the *Ashanti* and *Bemba* jointly. Every rhythm can be generated from one of these two by at most four swaps. More often than not, a *SplitsTree* will have additional nodes that do not correspond to any one of the input rhythms. Such nodes determine implied "ancestral" rhythms from which their "offspring" may be easily derived with the fewest number of swaps (mutations). Surprisingly, in this study the *SplitsTree* computed for the ten bell patterns using the swap distance measure yields no ancestral rhythms not contained in this set. However, the *Bemba* and *Yoruba* are tied for being the *minimal* generators of all the rhythms: both can generate all other rhythms with no more than 23 swaps.

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