

A Unified Theory of Proportion

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Abstract

Key elements of the systems of proportions used in ancient sacred geometry will be shown to be the ratio of diagonal to edge of various of regular n -gon. The diagonals are shown to be edges of various species of star polygons related to the n -gon. These diagonals or edges are shown to be the roots of two species of polynomials derived from Pascal's triangle and related to the Fibonacci and Lucas sequences when the n -gon has unit radius. All diagonals of regular n -gons are shown to have additive properties. Special attention is given to the heptagon.

1. Introduction

Through many cultures, star polygons were used as sacred symbols with the star of David and the Sri Yantra Hindu patterns as two examples. The fact that Venus traverses a five pointed star by making five closest approaches to Earth over an eight year cycle in the heavens as seen from the Earth was known to ancient civilizations. Also the designs of ancient sacred geometry use a small vocabulary of proportions

such as $\sqrt{2}$, $\sqrt{3}$, the golden mean $\tau = \frac{1 + \sqrt{5}}{2}$ and the silver mean $\theta = 1 + \sqrt{2}$. We will show that all of

these constants can be related to the edge lengths of star polygons and that they are ultimately related to a sequence of numbers called silver means the first of which is the golden mean. The geometry of the star heptagon will be found to be particularly interesting. We have previously shown that star polygons are also related to the chaotic dynamics of the logistic equation [1].

2. Star polygons

A regular star polygon is denoted by the symbol $\{n/k\}$ where n is the number of vertices and edges while k indicates that each vertex is connected to the k -th vertex from it in a clockwise direction. Under certain conditions a regular star polygon can be drawn in a single stroke such as polygon $\{5/2\}$ in Figure 1a or in more than one stroke as polygon $\{6/2\}$ in Figure 1b. Is there a simple rule that predicts whether the polygon is irreducible (can be drawn in a single stroke) or reducible (cannot be drawn in a single stroke)? The answer is simple: a polygon $\{n/k\}$ will be irreducible if and only if n and k are relatively prime.

Only connected stars are considered to be star polygons. For example, $\{6/2\}$ actually corresponds to the connected 3-gon $\{3/1\}$. As a result of the fact that n and k must be relatively prime, there are as many regular n -gons as there are positive integers relatively prime to n denoted by the Euler phi-function $\phi(n)$. For example, there are $\phi(7) = 6$ species of star 7-gons and $\phi(12) = 4$ species of 12-gons. The three clockwise oriented 7-gons are shown in Figure 2 with three identical but counterclockwise oriented 7-gons not shown. Of the 12-gons illustrated in Figure 3 only $\{12/1\}$ and $\{12/5\}$ are actual 12-gons. They are the two clockwise oriented 12-gons with the two counterclockwise 12-gons not shown.

The edges of a star n -gon are the *diagonals* of the regular n -gon. By determining the lengths of the $n-3$ diagonals that can be drawn from a given vertex of a regular n -gon, including the edge, we are also determining the lengths of the edges of various species of star n -gons. Furthermore, when all polygons are normalized to radius equal to one, the diagonals and edges of any m -gon reappear in n -gons whenever

n is a multiple of m. For example, in Figure 3 the edges of the triangle, square and hexagon appear as diagonals of a regular 12-gon. In this way we are able to determine all of the star m-gons related to an n-gon by considering all of the factors of n. For example, consider the 12-gon. Its *factor tree* is,

$$3 \rightarrow 6 \rightarrow 12 \leftarrow 4 \leftarrow 2. \quad (1)$$

The edges of the triangle reappear as diagonals of the hexagon and the 12-gon. Also the edges of the square reappear within the 12-gon. The factor 2 corresponds to the diameter $\{12/6\}$ in Figure 4 and can be thought of as a polygon with two edges, referred to as a *digon* $\{2/1\}$. In some limiting sense a single vertex can be thought to be a polygon with a single edge of zero length $\{1/1\}$. If we add up the total number of star polygons corresponding to these factors we get

$$\phi(1) + \phi(2) + \phi(3) + \phi(4) + \phi(6) + \phi(12) = 12. \quad (2)$$

These are all the star polygons related to the 12-gon. In addition to the single vertex and the digon, five of these have their edges oriented clockwise while five mirror images have retrograde edges. It can be shown that, in general,

$$\sum \phi(k) = n$$

where the summation is taken over the integers that divide evenly into n.

Star polygons are connected to number theory in many ways. One striking example is due to H.S.M. Coxeter in which the star polygon gives an elegant proof of Wilson's Theorem.

Wilson's Theorem: For p a prime number, $(p-1)! = -1 \pmod{p}$ or the integer $(p-1)! + 1$ is divisible by p.

Proof: For any p-gon there are clearly $(p-1)!$ factorial distinct p-gons. Since $\phi(p) = p-1$, p-1 of them are regular while the others are irregular. Since the regular polygons are invariant under rotation while the irregular ones are not, each irregular configuration gives a new polygon under rotation about the center through the angle between adjacent vertices. So if there are N classes of irregular polygons, when these are rotated about the p vertices, there are Np irregular polygons. Therefore,

$$(p-1)! = Np + p-1 \quad \text{or} \quad (p-1)! + 1 = (N+1)p.$$

And, as a result, $(p-1)! + 1$ is divisible by p or $(p-1)! = -1 \pmod{p}$.

3. Pascal's Law and Star Polygons

Consider Pascal's triangle.

Table 1 Pascal's Triangle

1	1	1				
1	1	1				
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

Starting from the left in Table 1 the diagonals: 111..., 123..., 136..., etc. appear as columns in which each successive column is displaced, in a downward direction, from the previous column by two rows. The rows of Table 2 can be seen to be the diagonals of Pascal's triangle related to the Fibonacci sequence with the sum of the elements of Row n being the n-th Fibonacci number. This table, referred to as the Fibonacci-Pascal Triangle or FPT, is associated with the coefficients of a series of polynomials, $F_1(n)$, (the superscripts are the exponents of the polynomials) [2].

Table 2. Fibonacci - Pascal Triangle

n/k	1	2	3	4	5	Sum	
0	1 ⁰					1	1
1	1 ¹					1	x
2	1 ²	1 ⁰				2	x ² + 1
3	1 ³	2 ¹				3	x ³ + 2x
4	1 ⁴	3 ²	1 ⁰			5	x ⁴ + 3x ² + 1
5	1 ⁵	4 ³	3 ¹			8	x ⁵ + 4x ³ + 3x
6	1 ⁶	5 ⁴	6 ²	1 ⁰		13	x ⁶ + 5x ⁴ + 6x ² + 1
7	1 ⁷	6 ⁵	10 ³	4 ¹		21	x ⁷ + 6x ⁵ + 10x ³ + 4x ¹
8	1 ⁸	7 ⁶	15 ⁴	10 ²	1 ⁰	34	x ⁸ + 7x ⁶ + 15x ⁴ + 10x ² + 1
						etc.	etc,

Each column in Table 2 begins with a 1. The numbers (not the exponents) are generated by the recursion relation:

$$(n,k) = (n-1,k) + (n-2,k-1)$$

where (n,k) denotes the number in the n-th row and k-th column. For example,

(7,3) = (6,3) + (5,2) or 10 = 6 + 4. Also, beginning with 1 and x, each Fibonacci polynomial $F_1(n)$ is gotten by multiplying the previous one, $F_1(n-1)$ by x and adding it to the one before it $F_1(n-2)$, e.g., $F_1(3) = x F_1(2) + F_1(1)$ or $x^3 + 2x = x(x^2 + 1) + x$.

Letting $x = 1$ in the polynomials yields the Fibonacci sequence:

1 1 2 3 5 8 ... The ratios of successive numbers in this series converge to the solution to $x - 1/x = 1$ or the golden mean τ which I shall also refer to as the *first silver mean of type 1* or $SM_1(1)$.

Letting $x = 2$ yields Pell's sequence (see Sec. 7.4):

1 2 5 12 29 70 ... (e.g., to get a number from this sequence, double the preceding term and add the one before it). The ratio of successive terms converges to the solution of $x - 1/x = 2$ or the number $\theta = 1 + \sqrt{2} = 2.414213...$, referred to as the silver mean or more specifically as the *2nd silver mean of type 1*, $SM_1(2)$.

Letting $x = 3$ yields the sequence: 1 3 10 33 109 ... (e.g., to get a number from the sequence, triple the preceding term and add the one before it). The ratio of successive terms converges to the solution to $x - 1/x = 3$ which is the *3rd Silver Mean of type 1* or $SM_1(3)$.

In general letting $x = N$, where N is either a positive or negative integer, leads to an approximate geometric sequence for which,

$$x_{k+1} = Nx_k + x_{k-1},$$

and whose ratio of successive terms is $SM_1(N)$ which satisfies the equation,

$$x - 1/x = N. \quad (3a)$$

When the polynomials in Table 2 have alternating signs they are denoted by $F_2(x)$, and upon letting $x = N$,

$$x_{k+1} = Nx_k - x_{k-1},$$

and whose ratio of successive terms is $SM_2(N)$, silver means of the second kind, which satisfies the equation,

$$x + 1/x = N. \quad (3b)$$

4. Lucas Version of Pascal's Triangle

Fibonacci and Lucas sequences are intimately connected. The standard Fibonacci sequence, $\{F_n\}$, is: 1 1 2 3 5 8 13 ... while the standard Lucas sequence, $\{L_n\}$, is 1 3 4 7 11 18 .. where $L_n = F_{n-1} + F_{n+1}$.

Adamson has discovered another variant of Pascal's triangle related to the Lucas sequence. In fact this Lucas - Pascal Triangle or LPT demonstrates that silver mean constants and sequences are part of an interrelated whole. Along with the FPT these tables are carriers of all of the significant properties of the silver means.

To construct the LPT, create a new "Pascal's triangle" with 1's along one edge and 2's along the other as shown in Table 3.

Table 3. A Generalized Pascal's Triangle

$$\begin{array}{ccccccc}
 & & & & & & 2 \\
 & & & & & & 2 & 1 \\
 & & & & & & 2 & 3 & 1 \\
 & & & & & & 2 & 5 & 4 & 1 \\
 & & & & & & 2 & 7 & 9 & 5 & 1 \\
 & & & & & & & & & & \dots
 \end{array}$$

As before each diagonal becomes a column of the LPT in which the elements in each successive column are displaced downwards by two rows. The exponents of the corresponding polynomials are sequenced as for the FPT. You will notice that the numbers in each row sum to the *Lucas sequence* and therefore I refer to the associated polynomials as Lucas polynomials $L_1(n)$ to distinguish these from Lucas polynomials with alternating signs denoted by $L_2(x)$.

Table 4. Lucas - Pascal's Triangle

n/k	1	2	3	4	Sum	
0	2 ⁰				2	2
1	1 ¹				1	x
2	1 ²	2 ⁰			3	x ² + 2
3	1 ³	3 ¹			4	x ³ + 3x
4	1 ⁴	4 ²	2 ⁰		7	x ⁴ + 4x ² + 2
5	1 ⁵	5 ³	5 ¹		11	x ⁵ + 5x ³ + 5x
6	1 ⁶	6 ⁴	9 ²	2 ⁰	18	x ⁶ + 6x ⁴ + 9x ² + 2
7	1 ⁷	7 ⁵	14 ³	7 ¹	29	x ⁷ + 7x ⁵ + 14x ³ + 7x
	etc.					etc.

Beginning with 2 and x, a Lucas polynomial is generated by the recursive formula :

$$L_1(n) = x L_1(n-1) + L_1(n-2),$$

for example $L_1(3) = xL_1(2) + L_1(1)$ or $x^3 + 3x = x(x^2 + 2) + x$.

Setting $x = 1, 2, 3, \dots$ in the Lucas polynomials generates a set of Generalized Lucas sequences that are related to the $SM_1(N)$ constants. Many interesting properties of these sequences are described in [1]. They also lead directly to an infinite set of generalized Mandelbrot sets [1].

5. The relationship between Fibonacci and Lucas polynomials and regular star polygons.

We have discovered a simple relationship between the roots of both the Fibonacci and Lucas polynomials with alternating signs and the diagonals of regular polygons when the *radii of the polygons are taken to be 1 unit*.

For odd n , the positive roots of the n -th *Lucas polynomial*, $L_2(n)$, with alternating signs equal the distinct diagonal lengths d_k for $k > 1$ and edge d_1 of the n -gon of radius 1 unit where

$$d_k = 2\sin k\pi/n \text{ for } k = 1, 2, \dots, (n-1)/2 \quad (4)$$

Example 1: For a pentagon, d_1 and $d_2 = \tau d_1$ where $d_1 = (\sqrt{(1 + \tau^2)})/\tau$ are the roots of $L_2(5)$:

$$x^5 - 5x^3 + 5x = 0,$$

This gives the familiar result that the ratio of the diagonal to the edge of a regular pentagon is the golden mean τ .

Example 2: For a heptagon, d_1 , $d_2 = \rho d_1$ and $d_3 = \sigma d_1$ where $d_1 = \sqrt{[(\rho\sigma - 1)/\rho\sigma]}$ are the roots of $L(7)$ where $\rho = 1.8019377\dots$ and $\sigma = 2.2469796\dots$:

$$x^7 - 7x^5 + 14x^3 - 7x = 0,$$

The numbers have additive properties much as the golden mean and this will be discussed in the next section.

For even n , the positive roots of the $(n-1)$ st *Fibonacci polynomial* with alternating signs $F_2(n-1)$, equal the lengths d_k of the distinct diagonals of regular n -gons of radius 1 unit where

$$d_k = 2\sin k\pi/n \text{ for } k = 1, 2, \dots, (n-2)/2. \quad (5)$$

For polygons with even n , one of the diagonals is twice the radius or 2. This diagonal is not one of the roots.

Example 3: For a hexagon, d_1 and $d_2 = \sqrt{3}d_1$ where $d_1 = 1$ are the roots of $F_2(5)$:

$$x^5 - 4x^3 + 3x = 0.$$

Example 4: For an octagon, d_1 , $d_2 = \sqrt{(\theta\sqrt{2})}d_1$ and $d_3 = \theta d_1$ where $d_1 = \sqrt{(\sqrt{2}/\theta)}$ are the roots of $F_2(7)$:

$$x^7 - 6x^5 + 10x^3 - 4x = 0.$$

The results for several polygons are summarized in Table 5. The diagonals are normalized to an edge value of 1 unit by dividing by d_1 .

Table 5. Lengths of Normalized Diagonals of n-gons

n-gon n	Lengths of normalized diagonals d_k/d_1					
	d_1	d_2/d_1	d_3/d_1	d_4/d_1	d_5/d_1	d_6/d_1
3	$\sqrt{3}$					
4	$\sqrt{2}$	$\sqrt{2}$				
5	$5^{1/4} \tau^{-1/2}$	τ				
6	1	$\sqrt{3}$	2			
7	$\sqrt{\frac{\rho\sigma - 1}{\rho\sigma}}$	ρ	σ			
8	$\sqrt{(\sqrt{2}/\theta)}$	$\sqrt{(\theta\sqrt{2})}$	θ	$\sqrt{(2\theta\sqrt{2})}$	2	
10	$1/\tau$	$5^{1/4} \tau^{1/2}$	τ^2	$5^{1/4} \tau^{3/2}$	2	
12	$\sqrt{(2-\sqrt{3})}$	$\sqrt{(2+\sqrt{3})}$	$\sqrt{2}\sqrt{(2+\sqrt{3})}$	$\sqrt{3}\sqrt{(2+\sqrt{3})}$	$2+\sqrt{3}$	2

We find the curious property that both the sum and product of the squares of the diagonals of an n-gon (including the edge) equals an integer and that this integer equals n for odd values of n. For example, $d_1^2 + d_2^2 = 5$ and $d_1^2 \times d_2^2 = 5$ for the pentagon, while, $d_1^2 + d_2^2 + d_3^2 = 7$ and $d_1^2 \times d_2^2 \times d_3^2 = 7$ for the heptagon.

Not only are the diagonals of regular polygons determined by Equations 4 and 5, but the areas A of the regular n-gons with unit radii are computed from the elegant formula,

$$A = \frac{n}{2} \sin \frac{2\pi}{n}$$

From this equation the square is found to have area 2 units while the 12-gon has area 3 units. It can also be determined that if n approaches infinity, then A approaches π , the area of a unit circle.

Notice that the key numbers in the systems of proportions based on various polygons present themselves in Table 5: τ -pentagonal system; θ and $\sqrt{2}$ - octagonal; $\sqrt{3}$, $1+\sqrt{3}$, and $2+\sqrt{3}$ - dodecahedral; ρ and σ - heptagonal, and these are pictured in Figure 6..

6. The Relationship between Number and the Geometry of Polygons

Since the unique diagonals of an n-gon correspond to the roots of a polynomial, the fact that these diagonals recur in any m-gon where n is a multiple of m serves to factor the polynomial into polynomials of smaller degree with integer coefficients. For example, the polynomial $F_2(9)$ of the 10-gon factors into the product of $L_2(5)$ of the 5-gon and $F_2(4)$, i.e., $F_2(9) = L_2(5) \times F_2(4)$ or,

$$x^9 - 8x^7 + 21x^5 - 20x^3 + 5x = (x^4 - 3x^2 + 1)(x^5 - 5x^3 + 5x).$$

We can state this result as a theorem:

Theorem 1: The polynomial $F_2(2n-1)$ of any $2n$ -gon factors into the product of $L_2(n)$ and $F_2(n-1)$.

By the same reasoning as for the 10-gon, the factor tree of Expression 1 can be used to factor $F_2(11)$, the polynomial representing the 12-gon. Of the six unique diagonals of the 12-gon, one occurs in the 3-gon (equilateral triangle), an additional one appears in the 6-gon (hexagon), another appears in the 4-gon (square), and two additional diagonals occur in the 12-gon. By Theorem 1, the polynomial of the 12-gon factors into,

$$F_2(11) = L_2(6) \times F_2(5)$$

corresponding to the factoring by the hexagon polynomial $F_2(5)$. The hexagon polynomial can then be decomposed further as,

$$F_2(5) = L_2(3) \times F_2(2)$$

corresponding to factoring by the triangle $L_2(3)$. These two factorizations can be combined to obtain,

$$F_2(11) = L_2(3) \times F_2(2) \times L_2(6)$$

or,
$$x^{11} - 10x^9 + 36x^7 - 56x^5 + 35x^3 - 6x = (x^3 - 3x)(x^2 - 1)(x^6 - 6x^4 + 9x^2 - 2)$$

Finally $L_2(6)$ factors into,

$$L_2(6) = (x^2 - 2)(x^4 - 4x^2 + 1).$$

The diagonal (edge) of the triangle comes from $L_2(3)$, the additional diagonal (edge) of the hexagon from $F_2(2)$, the diagonal of the square is the root of the first factor of $L_2(6)$ while the two additional diagonals of the 12-gon are the roots of the second factor of $L_2(6)$. Finally, the diagonal of the digon is the diameter of the 12-gon. This accounts for the six distinct diagonals of the 12-gon.

In what follows the symbol D_k will be used for diagonals of regular polygons *normalized to a unit edge* rather a unit radius.

7. Additive properties of the diagonal lengths

Similar to τ and θ , the diagonals of each of these systems of n -gons have additive properties. Steinbach has derived the following *Diagonal Product Formula (DPF)* that defines multiplication of the diagonal lengths in terms of their addition [3, 4],

$$D_h D_k = \sum_{i=0}^h D_{k-h+2i}, \text{ where } h \leq k.$$

where the diagonals have been normalized to polygons with edges of $D_1 = 1$ unit. It is helpful to write these identities in an array as follows:

$$\begin{array}{lll}
D_2^2 = 1 + D_3 & D_3^2 = 1 + D_3 + D_5 & D_4^2 = 1 + D_3 + D_5 + D_7 \quad (6) \\
D_2D_3 = D_2 + D_4 & D_3D_4 = D_2 + D_4 + D_6 & D_4D_5 = D_2 + D_4 + D_6 + D_8 \\
D_2D_4 = D_3 + D_5 & D_3D_5 = D_3 + D_5 + D_7 & \dots \\
D_2D_5 = D_4 + D_6 & \dots & \dots
\end{array}$$

These formulas are applied to the pentagon and the heptagon.

Example 1:

For the pentagon, $D_2 = D_3 = \tau$ and these relationships reduce to the single equation,

$$D_2^2 = 1 + D_3 \quad \text{or} \quad \tau^2 = 1 + \tau$$

Example 2:

The proportional system based on the heptagon is particularly interesting [4], [5]. For the heptagon, $D_2=D_5=\rho$ and $D_3=D_4=\sigma$ and these relationships reduce to the four equations,

$$\begin{array}{ll}
D_2^2 = 1 + D_3 & \text{or} \quad \rho^2 = 1 + \sigma \\
D_2D_3 = D_2 + D_4 & \text{or} \quad \rho\sigma = \rho + \sigma \\
D_3^2 = 1 + D_3 + D_5 & \text{or} \quad \sigma^2 = 1 + \sigma + \rho
\end{array}$$

What is astounding is that not only are the products of the diagonals expressible as sums but so are the quotients. Table 6 illustrates the quotient table for the heptagon.

Table 6 Ratio of diagonals (left /top)

	1	ρ	σ
1	1	$1 + \rho - \sigma$	$\sigma - \rho$
ρ	ρ	1	$\rho - 1$
σ	σ	$\sigma - 1$	1

As a result of DPF and the quotient laws, Steinbach has discovered that the edge lengths of each polygon form an algebraic system closed under the operations of addition, subtraction, multiplication, and division. Such algebraic systems are known as *fields* and he refers to them as *golden fields*.

8. The Heptagonal System

The heptagonal system is particularly rich in algebraic and geometric relationships. The additive properties of DPF and Table 6 for the heptagon are summarized:

$$\begin{array}{l}
\rho + \sigma = \rho\sigma \quad (\text{Compare this with } \tau + \tau^2 = \tau\tau^2) \\
1/\rho + 1/\sigma = 1 \quad (\text{Compare this with } 1/\tau + 1/\tau^2 = 1) \\
\rho^2 = 1 + \sigma \\
\sigma^2 = 1 + \rho + \sigma \\
\rho/\sigma = \rho - 1 \\
\sigma/\rho = \sigma - 1 \quad (7) \\
1/\sigma = \sigma - \rho \\
1/\rho = 1 + \rho - \sigma
\end{array}$$

The algebraic properties of each system of proportions are manifested within the segments of the n-pointed star (the n-polygon with all of its diagonals) corresponding to that system. For example, the pentagonal system of proportions is determined by the 5-star while the octagonal system is determined by the 8-star. Figure 5 illustrates the family of star heptagons. Notice that the short diagonal of length ρ (the edge is 1 unit) and the long diagonal of length σ are subdivided into the following segments depending on ρ and σ :

$$\begin{aligned} \rho &= 1/\rho + 1/\rho\sigma + 1/\sigma^2 + 1/\rho\sigma + 1/\rho \quad \text{and,} \\ \sigma &= 1/\sigma + \rho/\sigma^2 + 1/\sigma^2 + 1/\rho\sigma + 1/\sigma^2 + \rho/\sigma^2 + 1/\sigma \end{aligned}$$

Thus we see at the level of geometry that the graphic designer encounters the same rich set of relationships as does the mathematician at the level of symbols and algebra.

The following pair of intertwining geometric σ -sequences and corresponding Fibonacci-like integer series exhibit these additive properties:

$$\dots 1/\sigma \quad \rho/\sigma \quad 1 \quad \rho \quad \sigma \quad \sigma\rho \quad \sigma^2 \quad \sigma^2\rho \quad \sigma^3 \quad \sigma^3\rho \quad \sigma^4 \quad \dots \quad (8a)$$

$$1 \quad 1 \quad 1 \quad 2 \quad 3 \quad 5 \quad 6 \quad 11 \quad 14 \quad 25 \quad 31 \quad \dots \quad (8b)$$

The integer series is generated as follows:

- 1) Determine the first five terms $x_1x_2x_3x_4x_5$ beginning with 111
- 2) Let $x_2+x_1=x_4$ and $x_4+x_1=x_6$, i.e., $1+1=2$ and $2+1=3$ to obtain 11123
- 3) Repeat step 2 beginning with the $x_4x_5x_6$, i.e., from 123, $2+3=5$ and $5+1=6$ to obtain 12356
- 4) In general, for $n > 3$, $x_n = x_{n-2} + x_{n-1}$ and $x_{n+1} = x_{n-3} + x_n$.

The ratio of successive terms of this sequence equals, alternatively ρ and σ/ρ while the ratios of successive terms of the integer series asymptotically approaches ρ and σ/ρ , e.g., $25/14 = 1.785\dots \approx \rho$ while $31/25 = 1.24 \approx \sigma/\rho$. Also σ is obtained as the product of these ratios, i.e., $31/14 = 2.214 \approx \sigma$. Just as every power of the golden mean τ can be written as a linear combination of 1 and τ with the Fibonacci numbers as coefficients [6], every power of σ can be written as the following linear combinations of 1, ρ , σ where the integers of Sequence 8b appear as the coefficients:

$$\begin{aligned} \sigma &= 1\sigma + 0\rho + 0 \\ \sigma^2 &= 1\sigma + 1\rho + 1 \\ \sigma^3 &= 3\sigma + 2\rho + 1 \\ \sigma^4 &= 6\sigma + 5\rho + 3 \\ \sigma^5 &= 14\sigma + 11\rho + 6 \\ \sigma^6 &= 31\sigma + 25\rho + 14 \\ &\dots \end{aligned} \quad (9)$$

Notice that the first coefficient in the equation for σ^{n+1} is the sum of the three coefficients of the equation for σ^n while the second coefficient is the sum of the first two coefficients and the last coefficient is the same as the first of the previous equation, e.g., in the equation for σ^4 : $6=3+2+1$, $5=3+2$, and $3=3$.

A geometric analogy to the golden mean can be seen by considering the pair of rectangles of proportions $\rho:1$ and $\sigma:1$ in Figure 6. By removing a square from each, we are left in both cases with rectangles of proportion $\rho:\sigma$ although oriented differently.

9. Conclusion

With the aid of Pascal's triangle, the golden mean and Fibonacci sequences were generalized to a family of silver means. The Lucas sequence was then generalized with the aid of a close variant of the Pascal's triangle. These generalized golden means and generalized F- and L-sequences were shown to form a tightly knit family with many properties of number. Perhaps it is for this reason that they occur in many dynamical systems. We have shown that all systems of proportion are related to a set of polynomials derived from Pascal's triangle. These systems are related to both the edges of various species of regular star polygon and the diagonals of regular n -gons, and they share many of the additive properties of the golden mean. The heptagon was illustrated in detail.

Bibliography

- [1] J. Kappraff, *Beyond Measure: A Guided Tour through Nature, Myth, and Number*, Singapore: World Scientific Publ. (2002).
- [2] G.W. Adamson Private communication
- [3] P. Steinbach, Golden fields: A case for the heptagon, *Math. Mag.* 70(1)(1997).
- [4] P. Steinbach, Sections Beyond Golden, in *Bridges: 2000*, ed. R. Sarhangi, Winfield (KS): Central Plains Books (2000).
- [5] Ogawa, T., Generalization of the golden ratio: any regular polygon contains self-similarity, in *Research of Pattern Formation*, ed. R. Takaki, KTK Scientific Publ. (1990).
- [6] J. Kappraff, *Connections: The Geometric Bridge between Art and Science, 2nd ed.*, Singapore: World Scientific Publ. (2001).

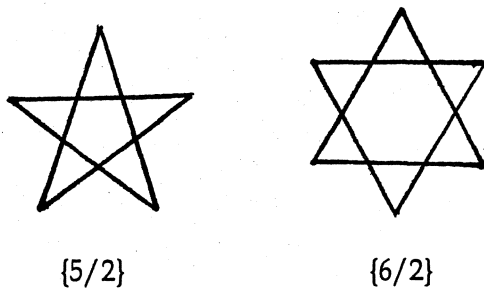


Figure 1. The $\{5/2\}$ star can be drawn in a single stroke; the $\{6/2\}$ star cannot be drawn in a single stroke.

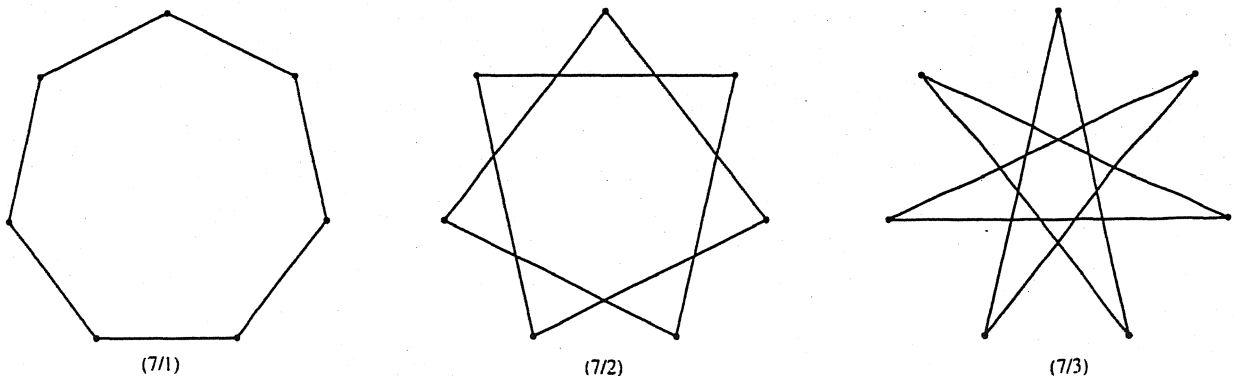


Figure 2. Three positively oriented 7-gons

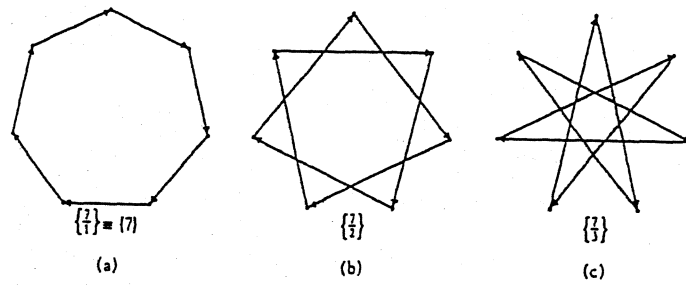


Figure 22.3 The three positively oriented star 7-gons.

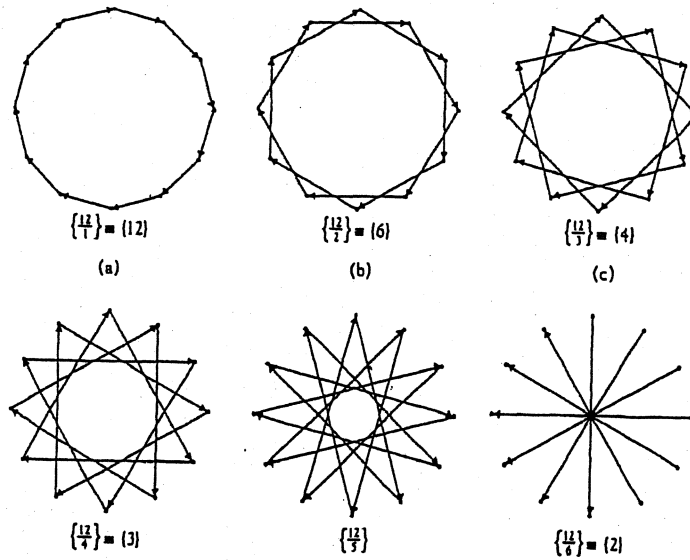
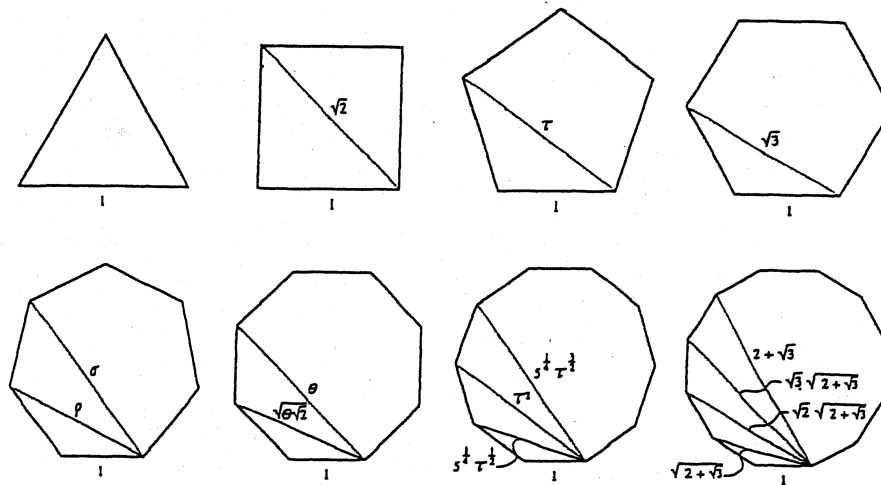


Figure 3. The family of star polygons related to the 12-gon. Note the small arrows indicating orientation.

Figure 4. The n-gons for n = 3, 4, 5, 6, 7, 8, 10, 12 with edge equal to 1 unit showing their diagonals.



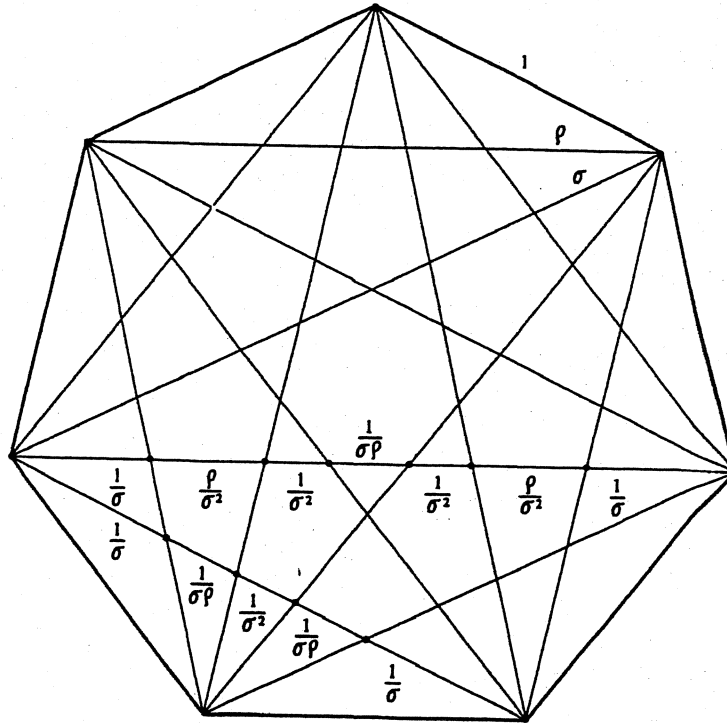


Figure 5. The diagonals of a heptagon subdivide themselves into lengths related to the lengths of the two principal diagonals ρ and σ .

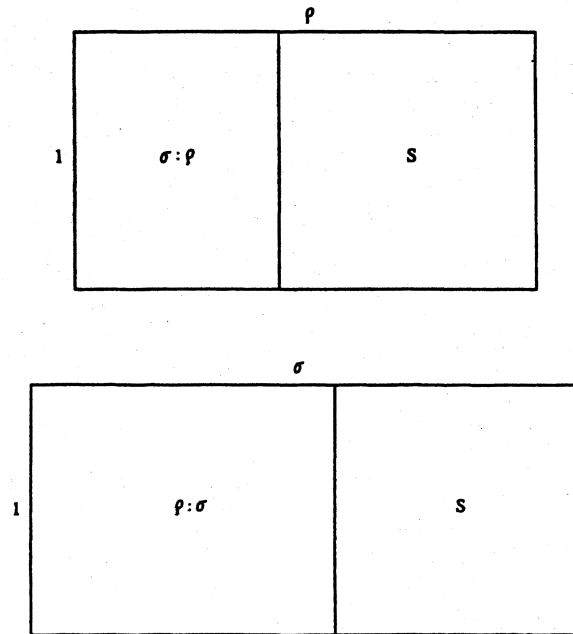


Figure 6. When a square is removed from rectangles of proportions $1: \rho$ and $1: \sigma$, rectangles of proportion $\rho:\sigma$ in different orientations remain.