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# Composable Art: Objects That Can Be Arranged in Many Ways

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**Abstract**

The concept of composable art is introduced and four examples of composable art objects are given. We analyze the number of different compositions that can be made for each object using combinatorics and the dynamic programming technique.

## 1 Introduction

Most traditional art is a painting or object created by an artist that is completely finished, in the sense that no changes can be made. Some forms of modern art are less rigid, like kinetic art. These include the mobiles of Alexander Calder and the mechanical installations of Jean Tinguely. One can also imagine art or design objects that allow various compositions to be made. The artist or designer chooses the limits within which variations are possible, and provides a mechanism by which different compositions can be made. Once a composition is made, the design object is in some fixed or stable configuration. We call such art or design objects composable art.

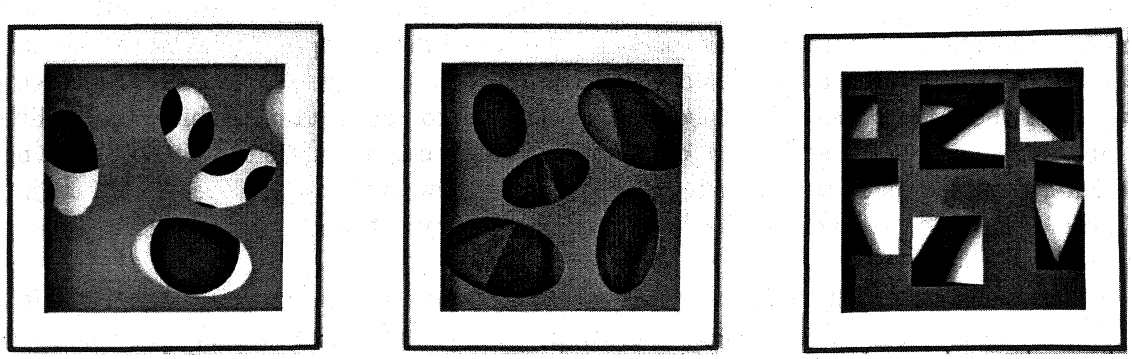

**Figure 1:** *Three compositions for the composable painting.*

Before proceeding with the introduction we give two examples. The *composable painting* consists of a box with a framed square opening in the front through which panels are visible. Each panel has a color and holes of a particular shape. Through the holes of one panel, the next panel behind it can be seen. Panels can be removed or inserted, and allow sliding over some distance in a slot. The concrete example has six panels of 50 by 100 cm of which four can be in the box at one time. One panel has square holes and is red, another has circular holes and is white, a third has triangular holes and is blue, etc. The outer box has a width of 155 cm; the square opening has size 45 by 45 cm. Figure 1 shows three compositions.

*Diago* has twenty wooden rods that come in five groups of four identical ones. The rods have a width of 27 mm and a thickness of 24 mm, except at regular intervals of 10 cm where the thickness is only 12 mm. The width of the thin part is 27 mm so that another rod fits if it is perpendicular. The connections between two rods that cross is basically the same as the logging cabin principle of stacking the beams. The shortest four rods have two thin parts, the next group of four has three thin parts, and likewise, there are four rods each with four, five, and six thin parts. The background panel is 150 cm by 150 cm and has nine protrusions that support the intersections of two rods.

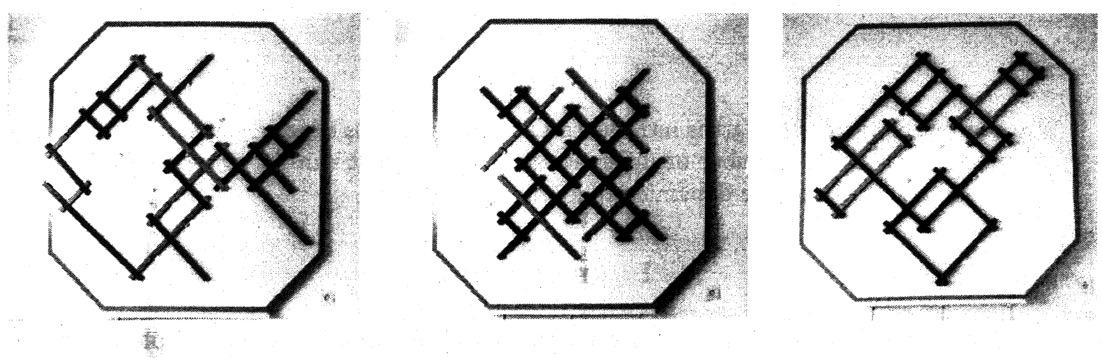

**Figure 2:** *Compositions for Diago.*

This paper gives two more examples of composable art and analyses the number of compositions that can be made with each of the four objects. We will use both discrete mathematics and small computer programs in the analysis. In many cases the exact number of compositions cannot be determined, which is due to the combination of two factors. Firstly, often there are physical limitations on particular compositions that are difficult to put into rules that would give regularity. Secondly, the number of compositions is often too large to generate all of them and test each one for the physical limitations. Another issue to note is that combinatorially different compositions do not necessarily have different visual appearance. In the composable painting, for instance, it can be that the first three panels block all view to the fourth one. So although there is a choice which panel to choose for the fourth one, it may not matter for the visual appearance of the object.

Composable art usually is abstract and geometric, and may involve symmetry. Since it must be possible to assemble different pieces in different ways, it is often the case that there is repetition of distances or lengths of sides. For example, the panels of the composable painting must be the same height so that all panels fit in all slots. More examples of composable art can be seen at `http://www.cs.uu.nl/~marc/composable-art/` .

Composability in art and design has been used before, but not widely, to the author's knowledge. The Swiss company Naef [8] makes design structures and toys, often with the idea of allowing many ways of arranging the pieces. Quite interesting is the Rietveld-Schröderhuis, located in Utrecht [7]. The upper level of this house has walls that can be slid or folded, so that either several rooms or a big open space appears. One could call it a composable house. Mozart wrote a musical piece consisting of parts that could be arranged in several ways. He suggested the use of dice to determine the actual piece [4]. Other related objects are construction toys, of which there are many examples: Lego, Meccano, Ministeck.

## 2 Combinatorics and dynamic programming

This paper gives the analysis of four composable art objects, all designed by the author. The analysis concerns the number of different compositions that can be made, assuming a suitable definition of what compositions are considered different. The analyses are based on standard techniques like binomial coefficients and recurrences. The recurrences are not of a standard form for which a closed solution exists, because certain conditions influence choices of the recurrence intermittently. Hence, a computer program can be used to count by using recursion. This implicitly generates all different composistions for the counting. However, it appears that the number of compositions is sometimes so large that this approach is infeasible. The remainder of this section describes how a technique called *dynamic programming* can be used to count more efficiently.

Dynamic programming was invented in the mid fifties by Bellman [2]. Currently, it is a standard algorithm design technique applied to optimization problems that have a *common sub-structure*, and is described in various textbooks on algorithms or optimization (e.g. [5]). It can be applied to problems related to travelling salesperson (bitonic Euclidean version), minimizing the penalty for extra spaces at the end of lines in a paragraph, optimal clustering problems, and many others.

Dynamic programming can also be used for counting problems. This is best illustrated by the computation of the Fibonacci numbers. Recall that the Fibonacci function is defined as $F(0) = 1$, $F(1) = 1$, and for all $i > 1$ we have $F(i) = F(i-1) + F(i-2)$. Assume we want to compute the $n$-th Fibonacci number. It is easy to implement a recursive function (or method in Java) that takes an integer $n$ and computes $F(n)$ by going into recursion twice if $n > 1$: once to determine $F(n-1)$ and once to determine $F(n-2)$. However, to compute $F(n)$ our procedure would be called $F(n)$ times, and as $F(n)$ is exponential in $n$, our solution has exponential running time as well (roughly proportial to $1.6^n$). Alternatively, we could compute $F(n)$ by using an array $A[0..n]$ and fill it from smallest to highest index. Each $A[i]$ with $i \geq 2$ is determined by looking up the (already computed and stored) values of $A[i-1]$ and $A[i-2]$. This way, much less time is needed to compute $F(n) = A[n]$ (roughly proportional to $n$ steps). For $n = 10$, the improvement is large, for $n = 30$ the improvement is dramatic. The use of a table to store solutions to subproblems that are needed many times in order to avoid re-computation is the main feature of dynamic programming.

## 3 Examples of composable art

**3.1. Composable painting.** A brief description of the composable painting was given in the introduction. All six panels are made of wood and have size 100 cm by 50 cm by 4 mm. One of the panels has no holes but has four areas of 50 by 50 cm in one color. It gives four possible background colors. The square opening in the outer box gives a view of 45 by 45 cm on the panels. The outer box is also made of wood and has size 155 cm by 55 cm by 8 cm approximately. The box is black inside, which gives a fifth possible background color. The box has four slots, each of which may fit one panel. Each panel can slide over 50 cm.

**Combinatorics:** We will analyze the possibilities when two or three panels with holes are used. There is a choice of panels, order, orientation of each panel, and sliding position of each panel. When using two panels with holes (or no background panel) there is also the choice which slot is left unused; it gives slightly different depth effects. We ignore this issue in the analysis.

It can easily be incorporated, however. We quantify sliding by assuming that each panel has 6 considerably different sliding positions, e.g. by a 10 cm sliding. Furthermore a panel can be placed in 4 orientations (exchange left-right and/or top-bottom). Therefore, the number of compositions for 2 panels is $\binom{5}{2} \cdot 2 \cdot 4^2 \cdot 6^2 \cdot 5 = 57,600$ because we choose 2 from 5 panels, the chosen panels can be placed in 2 orders, in 4 different orientations each, in 6 different sliding positions each, and there is choice of 5 background colors. Similarly, the number of compositions for 3 panels is $\binom{5}{3} \cdot 3! \cdot 4^3 \cdot 6^3 \cdot 5 = 4,147,200$.

**3.2. Composable tapestry.** The composable tapestry consists of 54 rods and one fixed piece at the top. The rods hang from each other and from the top piece by hooks. The rods have lengths 5, 45, 85, and 125 cm, and have, respectively, 1, 2, 3, and 4 hooks at the top and the same number at the bottom. The vertical distance between two rods is roughly 5 cm. In principle, all rods hang horizontally. The rods come in three groups. One group has aluminum rods with a square cross-section. The second group has transparent finished wooden rods with a round cross-section. The third group has black wooden rods with a diamond shaped cross-section (rotated square). Within each group, there are 9 pieces of 5 cm, and 3 pieces each of 45, 85, and 125 cm.

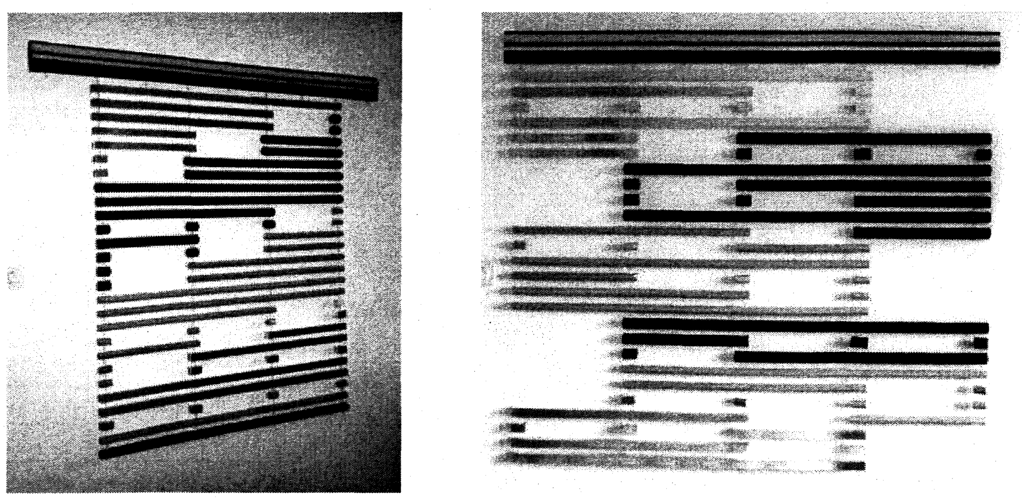

**Figure 3:** *Compositions for the composable tapestry.*

**Possible compositions:** A standard composition is one where 4 hooks of the top piece are used, and all rods fill a rectangular area of 125 cm wide and roughly 135 cm high (27 layers). It is possible to leave bigger holes, in which case more than 27 layers may appear. It is also possible to have rods stick out to the side, but in these cases balance comes into play. Rods could start to rotate away from the horizontal position. It is also possible to start with five hooks at the top piece and create wider compositions. Furthermore, one can hang some rods slightly non-horizontally by having the left and right sides at different layers. It is aesthetically better to have some degree of grouping rods of the same shape and color.

**Combinatorics:** There are an enormous number of compositions possible with the composable tapestry. We only analyze the standard compositions of 4 hooks wide and 27 layers high. We first count the number color assignments that are possible, and then we analyze the number of silhouettes where all rods are assumed to have the same cross-sectional shape and color. These quantities can be counted independently and multiplied for the total number of compositions.

Counting color assignments is easy. Any composition using all pieces will have 27 short pieces, 9 of each group, which are interchangeable. So there are $\binom{27}{9}$ times $\binom{18}{9}$ color assignments for the short rods, and similarly there are $\binom{9}{3}$ times $\binom{6}{3}$ color assignments for the 45 cm rods, and also for the 85 and 125 cm rods.

Counting silhouettes is considerably harder. By the restrictions of the hangings we consider, each layer will have either one 125 cm rod, or one 85 cm rod and a 5 cm rod, or two 45 cm rods, or one 45 cm rod and two 5 cm rods, or four 5 cm rods. If we denote by $A(j,k,m,n)$ the number of possible hangings with $j$ 125 cm rods, $k$ 85 cm rods, $m$ 45 cm rods, and $n$ 5 cm rods, then we get the following recurrence:

$$\begin{aligned}A(j,k,m,n) = \; & A(j-1,k,m,n) + 2\cdot A(j,k-1,m,n-1) + A(j,k,m-2,n) + \\ & 3\cdot A(j,k,m-1,n-2) + A(j,k,m,n-4)\end{aligned}$$

The factor 2 is needed to account for the possibilities that the 5 cm rod can be left or right of the 85 cm rod; the factor 3 is obtained in the same way. If any argument of $A(.,.,.,.)$ is negative, then $A$ is defined to be zero. We are interested in the value of $A(9,9,9,27)$. We could use a recursive computer program to count the number, but the program would make more than $7\cdot 10^{14}$) recursive calls (the value of $B(9,9,9,27)$, where $B$ is defined like $A$ but without the factors 2 and 3). Instead we use dynamic programming with a 4-dimensional array. This prevents the use of recursion if the same subproblem has been counted before. For the array, at most 28,000 values need be determined. We obtain $4.8\cdot 10^{19}$ compositions. For completeness, we list more values of $A(.,.,.,.)$ in Table 1.

| $A(1,1,1,3)$ | $A(2,2,2,6)$ | $A(3,3,3,9)$ | $A(4,4,4,12)$ | $A(5,5,5,15)$ |
|---|---|---|---|---|
| 36 | 3960 | 604,800 | $1.1\cdot 10^8$ | $2.1\cdot 10^{10}$ |

| $A(6,6,6,18)$ | $A(7,7,7,21)$ | $A(8,8,8,24)$ | $A(9,9,9,27)$ |
|---|---|---|---|
| $4.4\cdot 10^{12}$ | $9.4\cdot 10^{14}$ | $2.1\cdot 10^{17}$ | $4.8\cdot 10^{19}$ |

**Table 1:** *Number of silhouettes of the composable tapestry for different numbers of pieces.*

**3.3. Diago.** A description of Diago was given in the introduction.

**Possible compositions:** Diago allows all compositions where the rods are, effectively, placed on a grid, rotated by 45 degrees. Any composition where two parallel rods do not use the same grid position is possible. Two orthogonal rods can always use the same grid position. Since there are four rods of each length, various symmetric compositions are possible, including rotational symmetry over 90 or 180 degrees, and line symmetries in one or two lines. One "boundary condition" exists: all of the rods must be supported by the protrusions, directly or indirectly.

**Combinatorics:** The combinatorially distinct compositions that can be made with Diago can be counted in much the same way as with the composable tapestry, if we ignore the boundary condition. For convenience we rotate the back panel by 45 degrees and have horizontally and vertically placed rods. We first analyze in how many ways the rods can be placed only horizontally on a square grid. To this end we define $A(i, j, k, m, n, r, c)$ as the number of ways to place $i, j, k, m, n$ rods of lengths 2, 3, 4, 5, and 6, respectively, starting at row $r$ and column $c$ on a grid of size $[0..Z-1] \times [0..Z-1]$. We have:

$$A(i, j, k, m, n, r, c) = \sum \begin{cases} \text{number of ways assuming we place a 2-rod at } (r, c) \\ \text{number of ways assuming we place a 3-rod at } (r, c) \\ \text{number of ways assuming we place a 4-rod at } (r, c) \\ \text{number of ways assuming we place a 5-rod at } (r, c) \\ \text{number of ways assuming we place a 6-rod at } (r, c) \\ \text{number of ways assuming we leave } (r, c) \text{ empty.} \end{cases}$$

We can only place a 2-rod if $i > 0$ and $c \leq Z-2$, and then the number of ways is $A(i-1, j, k, m, n, r, c+2)$ or $A(i-1, j, k, m, n, 0, r+1)$, depending on whether the 2-rod fills up the row $r$ or not. If the 2-rod fills up the last row, then $A(1, 0, 0, 0, 0, Z-2, Z-1) = 1$, and $A(1, j, k, m, n, Z-2, Z-1) = 0$ if any of $j, k, m, n$ is positive. This is done to count only the compositions where all rods are used. In a similar way we can determine the number of ways when a 3-rod, a 4-rod, a 5-rod, or a 6-rod is placed, or no rod is placed at $(r, c)$. Using dynamic programming with a 7-dimensional array of size $5 \times 5 \times 5 \times 5 \times 5 \times Z \times Z$ we determine $A(i, j, k, m, n, 0, 0)$ for some choice of grid size $Z$ and for every subset of the 20 rods in total. There are $5^5 = 3125$ different subsets.

To determine the total number of compositions we consider all 3125 choices of using $i, j, k, m, n$ rods of each size horizontally and $4-i, 4-j, 4-k, 4-m, 4-n$ rods vertically. We get:

$$\text{No. of compositions} = \sum_{0 \leq i,j,k,m,n \leq 4} A(i, j, k, m, n, 0, 0) \cdot A(4-i, 4-j, 4-k, 4-m, 4-n, 0, 0)$$

| $6 \times 6$ | $7 \times 7$ | $8 \times 8$ | $9 \times 9$ | $10 \times 10$ | $11 \times 11$ | $12 \times 12$ | $13 \times 13$ | $14 \times 14$ |
|---|---|---|---|---|---|---|---|---|
| 0 | $2.1 \cdot 10^{16}$ | $6.5 \cdot 10^{23}$ | $3.1 \cdot 10^{28}$ | $9.9 \cdot 10^{31}$ | $6.7 \cdot 10^{34}$ | $1.6 \cdot 10^{37}$ | $1.8 \cdot 10^{39}$ | $1.1 \cdot 10^{41}$ |

**Table 2:** *Number of compositions of Diago for various grid sizes.*

The number of compositions for different grid sizes is shown in Table 2. We must remark that not all compositions we count are connected, and many cannot actually hang on the background panel without sliding off. Exhaustive enumeration and testing each composition appears to be impossible within a reasonable amount of time.

| $6 \times 6$ | $8 \times 8$ | $10 \times 10$ | $12 \times 12$ | $14 \times 14$ |
|---|---|---|---|---|
| 0 | 91,344 | $7.3 \cdot 10^6$ | $1.3 \cdot 10^8$ | $1.1 \cdot 10^9$ |

**Table 3:** *Number of symmetric compositions of Diago for various grid sizes.*

For symmetric compositions—90 degrees rotation or in 2 diagonal lines—we take one rod of each length and consider the number of ways the 5 rods can be placed horizontally on a grid of size $r \times (r/2)$ if $r$ is even. This specifies the placement of all other rods too by the symmetries, and no conflicts in positioning can occur. The number of such symmetric compositions is shown in Table 3. Similar ideas apply to odd values of $r$ and to different symmetries.

**Further analysis for Diago:** One deficiency in the analysis is that it does not take into account the nine protrusions that must support the composition. This condition does not combine well with the recurrence, and the whole analysis breaks down. Similarly, it is not possible to only count the compositions in which the rods form only one connected component. One possible way to estimate the number of compositions that are properly supported, or connected, or both, is Monte Carlo simulation. For a given composition we can easily program a test to determine if it is properly supported and/or connected. The array produced for dynamic programming allows us to generate a random composition, because the values in the array provides the distribution of counts for the subcompositions. The idea of using the dynamic programming tables for random generation has been observed before to generate random strings [6], random alignments of strings [1], and random outerplanar graphs [3]. If we generate 10,000 random compositions on an $8 \times 8$ grid and would find that, say, 21% is supported, then we could estimate the total number of compositions that are supported to be 21% of $6.5 \cdot 10^{23}$.

**3.4. Town and Forest.** The base is the only fixed piece of the Town and Forest. It is a shelf with three slots in the length direction. Since the sides of the shelf are slanting, the slots have slightly different lengths, namely 160, 170, and 180 cm from front to back. The slots have a width of 5 mm and fit the 12 pieces of 4 mm thick hardboard. There are 6 triangular pieces that are like pine trees. They have three different sizes and three different shades of green, which makes them all distinct. Furthermore, there are 4 pentagons that are like houses and a tower. The 11th piece is a hexagon and is a bigger building. The 12th piece is a trapezoid that can be considered a train. Some pieces have the same width. Two of the houses are the same. All other pieces are unique. The occurring widths are 36 cm, 35 cm, 29 cm, 27 cm (twice), 23 cm (three times), and 15 cm (four times).

**Possible compositions:** The 12 pieces fit in the three slots behind or next to each other. All pieces are widest at their base; there are no overhangs. Not all pieces fit in the same slot. To determine if a subset of the pieces fits in some slot we only have to add the base widths of the pieces and compare it to the slot length.

**Combinatorics:** Assume we have $k$ pieces $a_1, \ldots, a_k$ of width $|a_i|$. There are three slots, a front, middle, and back slot. Two compositions are the same if and only if they have the same pieces in the same slots and in the same order. We disregard the precise positions altogether and only consider combinatorially distinct compositions.

Assume first that the three slots are long enough to hold all pieces. If there were only one slot, the number of compositions would be $N_1(k) = k!$ . When there are two slots, we get:

$$N_2(k) = \sum_{i=0}^{k} \binom{k}{i} \cdot i! \cdot N_1(k-i) \;=\; \sum_{i=0}^{k} k! \;=\; (k+1)! \,,$$

(choose $i$ from $k$ pieces for the second slot, which can be placed in $i!$ orders). More generally, for $k$ pieces and $j$ slots, we can model all compositions by considering all $k!$ permutations and all ways of separating the sequence into $j$ groups (slots $1, \ldots, j$) by $j-1$ separators. This can be done in $(k+j-1)!/(j-1)!$ ways. Division by $(j-1)!$ is needed because all separators are the same for the partitioning.

For the Town and Forest, there are 12 pieces and 2 of them are the same. For computer-assisted counting, we can enumerate all possibilities and test if—in a slot assignment—the pieces fit. We only enumerate the $3^{12} = 531,441$ assignments into sets $S_1$, $S_2$, and $S_3$, one for each slot. If the subsets fit in their slot, that is, if

$$\sum_{a \in S_i} |a| \leq \text{ slot length}$$

for $i = 1, 2, 3$, then we count $|S_1|! \cdot |S_2|! \cdot |S_3|!$. A computer program counted 471,902 slot assignments that would fit, and counted 19,497,542,400 compositions. Since two pieces are the same, this last number must be halved.

## 4 Future work

We presented four composable art objects and analyzed the number of different hangings or compositions. The computer-assisted analysis of two of the objects was based on an algorithms design technique called dynamic programming. It allows to count the number of hangings without explicitly enumerating all. The technique is general; it can also be applied to another, quite different composable art object currently under construction. The other two composable art objects that were analyzed required straightforward combinatorics, one complemented with a simple implementation.

Other composable art objects and larger pictures of the ones described here can be seen at `http://www.cs.uu.nl/~marc/composable-art/` . Other designs are under construction or planned in the future. Besides the use of programs for combinatorics, it is also planned to develop software to generate and evaluate particular compositions. This requires a suitable definition of an aesthetical composition.

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