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Exploring Fractal Geometry in Music

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Abstract

From graphic design to stock market predictions, applications of fractal geometry permeate fields in every way. The depiction of fractals also has substantial artistic intrigue. Mathematical graphs displaying multiple dimensions of self-similarity reveal highly evocative perceptions of depth and design. Such a fascination in sight can also be perceived in sound. This research explains the depiction of fractal geometry in music. It examines various compositional techniques used to employ the representation of fractal symmetry in music. Upon considering such representations entailed in the compositional process and realized in the structure of a musical work, we use our findings to engineer an original method that musically depicts the geometry of a fractal. This method has its ultimate realization in a musical composition that amalgamates our discoveries. This paper will discuss fractal representations in music and the compositional procedures that achieved them.

1. Introduction

Fractal geometry, popularized by Benoit Mandelbrot [4] in the nineteen-seventies, remains a highly evocative topic. The most vivid example of a fractal is the Koch snowflake, constructed by the Swedish mathematician Helge von Koch in 1904. To begin construction of the Koch snowflake, start with an equilateral triangle of unit side. We call this starting figure the initiator. Next, to generate the figure, replace the middle third of each line segment of the triangle with two new line segments, each having length one-third of the original line segment. Each generation stage of the snowflake proceeds similarly, replacing each line segment with a copy of the generator, such that the newly adjoined line segments are one-third the length of those in the preceding stage. The snowflake has its ultimate realization by continuing its construction ad infinitum.

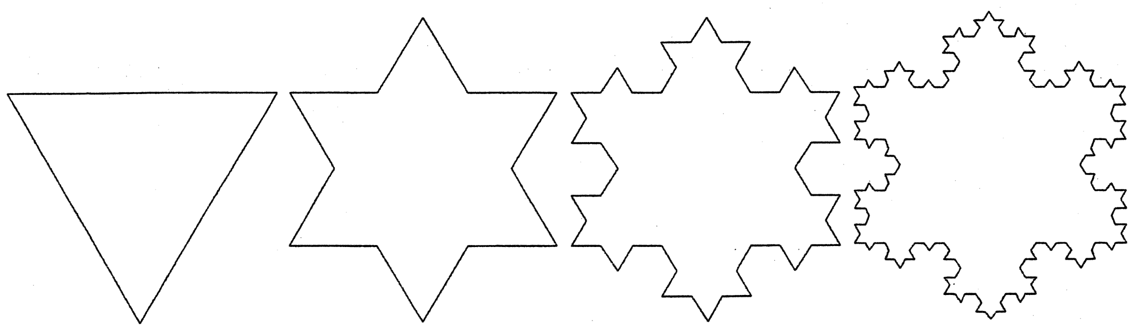

Figure 1. *The Koch Snowflake.*

Each side of the snowflake is self-similar. That is, if we were to zoom in on a given side any number of times, the resulting perspective would be identical to the original. The model of the Koch snowflake gives us a vivid conception of what a fractal is.

Fractals are not exclusive to figures such as the Koch snowflake. Self-similar fractals can be generated many ways and are found in many different areas. Yet, common to all is the relationship between the initiator and generator. For example, we can initiate and generate number sequences that become fractals. To begin, consider the set of integers S0 = {0,2,3}. Let S0 be our initiator, and to generate S0, simply add the entire set to each of its elements. The following diagram shows the first and second generations:

S0	0									2									3								
S1	0			2			3			2			4			5			3			5			6		
S2	0	2	3	2	4	5	3	5	6	2	4	5	4	6	7	5	7	8	3	5	6	5	7	8	6	8	9

Figure 2. *S0 and its First Two Generations.*

Continuing the generations indefinitely produces a number sequence with multiple dimensions of self-similarity. We can see this by removing integers from the sequence. If we remove every integer except every third or ninth, the previous sequence results. To generalize, the removal of every integer save every 3^kth integer results in the appearance of the previous sequence. Thus, the number sequence is embedded within itself on a multitude of different levels yielding a fractal structure. Fractal generation of number sequences such as this becomes particularly important when we want to create fractals in music. Before we learn how, we must discuss another aspect of fractals, *fractal dimension.*

1.1 Fractal Dimension. The fractal dimension tells us how densely a particular geometry occupies a given space. The calculation of a fractal's dimension allows us to objectively classify, compare, and contrast any number of fractals. One way of calculating fractal dimension is through the recognition of the affine self-similarity of an object. For an object to be affine self-similar, it must consist of congruent subsets, each of which can be magnified by a constant factor to yield the original object. To calculate the fractal dimension of such an object we have the following definition:

Definition: [2] Suppose the affine self-similar set S may be subdivided into k congruent pieces, each of which may be magnified by a factor of M to yield the whole set S. Then the fractal dimension D of S is

$$D = \log(k) / \log(M) = \log(\text{number of pieces}) / \log(\text{magnification factor}).$$

Comparing these qualities to the Koch snowflake, we find that the sides of the snowflake are affine self-similar. Thus, to calculate each side of the object, recall that each original side was decomposed into four smaller pieces with a magnification factor of 3. Applying our formula yields:

$$D = \log(4) / \log(3) = 1.2618\ldots$$

We only examine the sides of the triangle because no single side can be magnified to represent the entire object. Thus, the snowflake in its entirety is a composite of three Koch curves, each with the same dimension. Because of this, we can apply the following theorem:

THEOREM: [1] If dim(A) ≦ dim(B) for sets A and B, the dim(A U B) = dim(A).

Therefore labeling the sides of the Koch snowflake A, B, and C respectively and applying the theorem, we know dim(A) = dim(B) = dim(C), and the fractal dimension of the Koch snowflake is 1.2618.

2. Numbers to Notes

Mathematical influence has profoundly shaped how composers and theorists have advanced musicianship. Composers such as Pierre Boulez and Iannis Xenakis have musically implemented concepts such as algebraic structures and statistics. Recently, the intricate structures of fractal geometry have sparked new imagination in the construction of music. The resulting music depicting such geometry has been labeled "fractal music." **Fractal music** is a composition conceived and constructed based on the principles and applications of a specific type of fractal geometry that ultimately represents a fractal-like structure. Much of the existing fractal music is constructed via computer and results from the application of mathematical algorithms. However, fractals are conceived, generated, and depicted in music in many ways. Common to all, the quality that merits a composition's fractal nature is the presence of multiple dimensions of self-similarity (the whole is the part and the part is the whole).

We engineered an original method to compose a fractal music composition that fits our description. The composition in discussion, "Iterations I" for flute and piano, by Brian Hansen, was constructed algorithmically and without the aid of the computer. However, in order to fully realize a fractal composition we must consider if algorithmic application accomplishes our goal.

(a) Does the mathematical algorithm itself contain fractal geometry?

(b) Does the musical application of the fractal preserve the geometry?

(c) Does the realized composition reflect/preserve the geometry (does it depict self-similar dimensions)?

To answer these questions, we will discuss the application of the particular algorithmic method used to generate the fractal music of "Iterations I." We will also discuss how ultimately the composition contains self-similar dimensions and thus represents a fractal.

2.1 The Equal-Tempered Chromatic Scale. To compose a piece of music by algorithmic application, we will correspond integers to pitches in music. We will accomplish this correspondence using the equal-tempered chromatic scale. The chromatic scale consists of twelve different pitches. The pitches are assigned letter names starting with A and progressing to G. Of course these letters only account for seven of the twelve pitches, so what about the other five? The remaining five pitches lie between the lettered pitches and are designated as either sharp (#) or flat (b) relative the nearest lettered pitch. Thus, the pitches in a twelve-tone scale may appear sequentially two ways as:

Figure 3. *The Twelve-Tone Western Scale*.

It is important to know about interval content (distance between two pitches) in the chromatic scale. We can determine this by counting the number of "half-steps" or "whole-steps" between two pitches. A half-step is the distance between a pitch and the adjacent pitch on either side. For example, the pitches A and B are adjacent to Bb, thus these pitches are a half step apart. A whole step is the distance between a pitch and the second tone on either side of it. The second tones on both sides of C are D and Bb, so these pitches differ by whole-steps.

2.2 "Iterations I." The composition "Iterations I" for flute and piano, by Brian Hansen, is a musical depiction of the iterative process undergone using an initiator and generator to create a fractal. This particular fractal depiction was accomplished using number sequences. Recall the number sequences constructed using the initial set S0 = {0,2,3} and its two generations:

S0 = {0,2,3}
S1 = {0,2,3,2,4,5,3,5,6}
S2 = {0,2,3,2,3,5,3,5,6,2,4,5,4,6,7,5,7,8,3,5,6,5,7,8,6,8,9}.

These sequences have the potential for musical application. To accomplish this, simply assign numeric values to notes in the chromatic scale. In addition, since the scale includes only 12 tones, the sequence needs to be in mod 12. Now integer values can correspond to musical tones. For example, in "Iterations I" we chose the initial pitch as Bb, which corresponds to the integer 0. Next, we enumerate each pitch ascending stepwise from Bb to A.

Figure 4. *Integers Assigned to the Twelve-Tone chromatic Scale*.

Then, we can correspond the initial set S0 = {0,2,3} and its generations with the twelve-tone scale. For example, the initial set S0 = {0,2,3} corresponds with the pitches Bb, C, and Db respectively. We can continue this process with the ensuing generations of S0. The result is a series of pitch groups yielding a sonic representation of the numeric sequence. The following diagram shows S0 and its first two generations corresponded to notes.

Figure 5. *Correspondence between Number Sequences and Pitches.*

Upon examining the pitch sequences, we see that self-similarity occurs among the numerous three-pitch sets that spawn from the initial set {0,2,3} corresponding to {Bb, C, Db}. This self-similarity lies in the interval composition of three-pitch subgroups within the sequences. For example, the interval difference between C and Bb is a whole step, between Db and C is a half step. Thus, the interval composition of S0 is a whole step followed by a half step. Upon examining the pitch sequences, we see that every third pitch preserves this interval relationship. If we were to reconstruct our sequence by only including every third pitch, the resulting sequence would be the same as the previous. In fact, if our sequence consisted of only every 3^kth pitch, the previous sequence would result. This is the same multi-dimensional quality displayed by the fractal number sequence. Clearly, the musical application of the number sequence preserves its fractal quality.

To ultimately realize a composition, we must think about how our process incorporates into the many different elements of musical construction. To make music we need rhythm (pitch duration), dynamics (loudness), and form (framework). How can we use these elements to create music that depicts the fractal we constructed? After all, we have only conjured up a bunch of tones, and we must figure out how they meaningfully organize into a piece of music that exhibits fractal qualities. "Iterations I" attempts to accomplish this.

The structure of "Iterations I" outlines the devised iteration process. As the iteration process unfolds, the quantity of pitches increases and thus the activity and complexity of the music increases accordingly. The piece begins with solo flute announcing the initial set, S0 = {0, 2, 3} = {Bb, C, Db}, followed by the first and second iteration generations. Just before the flute enters the third iteration generation, the piano makes its introduction as the accompaniment, independently reinitiating the initial pitch set S0. The piano then provides an accompaniment basis for the flute with the ensuing generations of our initiator. The following diagrams show how flute and piano introduce the pitch sequences:

Figure 6. *Top: Flute Introducing S0 and S1. Bottom: Piano Introducing S0 and S1.*

At this point, we can detect the composition's depiction of self-similar dimensions. The flute introduces S0 and S1 melodically (one pitch at a time) in an ascending fashion. We notice the piano also introduces S0 melodically, however the pitches descend, are in a much lower register, and are temporally augmented. In addition, the piano presents the ensuing generation S1 harmonically (many pitches at once). These various depictions of our pitch generations give the listener alternate sonic perceptions of the sequences. The listener is hearing different layers of musical activity, which is perceived as multiple-dimensions of sound. The self-similarity is a product of the sonic integrity of S0 and its generations. Although the pitches are presented melodically, harmonically, at different times, speeds, and in different registers, the various techniques still produce a common sonority or harmonic quality. This is due to the integrity of three-pitch subgroups embedded in the constructed pitch sequences. Within each subgroup, the intervals between pitches remain constant. Thus, the structural similarity is sonically perceived. These multiple dimensions of self-similar sound reflect the geometry of a fractal.

The music ensues with dialogue between the duo until flute and piano exhaust the pitch material of the third iteration group. At this point, a small interlude of solo piano occurs leading to the next section. Then, drawing from the pitch material of the fourth and final iteration generations the flute and piano enter a development section that leads to a climax. The fourth generation yields the most pitch material, enabling the creation of the greatest musical activity and justifying the climactic atmosphere of the section. Upon the arrival of the climax, the flute exhausts most of its pitch material from the fourth iteration group, and the piano uses all. The climax is of particular importance, for it displays the depiction of the three-pitch subgroups in greatest variety:

Figure 7. *Climax Measures from "Iterations I".*

Notice the boxes outlining some of the harmonic and melodic portrayals of the three-pitch groups layered at different times, different speeds, and in different registers. This shows how the music itself depicts the geometry of our generated fractal and creates an aesthetic of an atmosphere with multiple dimensions of sound.

"Iterations I" takes us on a journey, showing us the compounding development of our iteration procedure. The composition structurally and sonically depicts multiple levels of self-similarity through use of the three-pitch subgroups embedded in the fractal pitch sequences. These subgroups constitute the entirety of melodic and harmonic material, whereby through contrapuntal layering of the subgroups at different times and speeds, a sonic texture of multi-dimensional depth is perceived. This multi-dimensional atmosphere is precisely what one would expect when encountering a fractal.

2.3 Geometry and Dimension for "Iterations I." Since "Iterations I" contains fractal structures, we should be able to determine its fractal dimension. Our sequence applied to the chromatic scale is analogous to sides on the Koch snowflake. The presence of only twelve pitches in the system is analogous to saying there are twelve "sides" to the music. Thus, the geometry of our initiator is a dodecagon. The generator spawns three elements (sides) upon each side of the initiator, and the process continues in subsequent generations replacing each side with a copy of the generator. By recognition of the sequence in Mod 12 and correlating this to a vector space representation akin to the Koch snowflake, we are able to graphically represent the fractal pitch sequence of "Iterations I"[5].

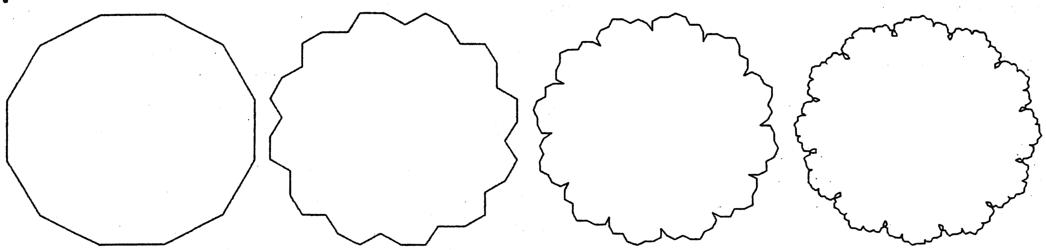

Figure 8: *The "Iterations I" Snowflake.*

To calculate fractal dimension, we know the generator of the dodecagon consists of three elements. The generator replaces each side of the figure with sides that are 1/2.39417 times the preceding side. This yields a magnification factor of approximately 2.39417. Applying this to the definition of fractal dimension yields:

$$D = \log(k) / \log(M) = \log(\text{number of pieces}) / \log(\text{magnification factor})$$
$$= \log(3) / \log(2.39417) = 1.25838....$$

Finally, to apply Theorem [1], again treat each pitch of the chromatic scale as one side. Then dim(A) = dim(B) = dim(C) = ... = dim(L). Therefore, the fractal dimension of our pitch sequence is 1.25838.

3. Conclusion

Fractals are highly evocative structures in their depiction of multi-dimensional depth and design. Their role in music has yet to be determined, but the application of fractal geometry is expanding new grounds for music. The composition "Iterations I" utilizes fractal number sequences and relates them to pitches, showing one of the many potential applications fractals can have in music. Still, not all doors have been opened, and the potential for depicting fractals in music is as great as the imaginations of artists that wish to achieve them.

References

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