Bridges
Mathematical Connections
in Art, Music, and Science

The Rootsellers —Retelling the Galois Group of a Quartic Polynomial

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Abstract

Two classroom puppet shows, offered to students from the third grade through college, discuss simple aspects of a deep result in group theory: properties of the Galois group of a polynomial. In order to convey the artistic component for this account, diagrams of mathematical symmetry and algebraic structure with fanciful annotations illustrate the lessons of the shows. Readers may find resonance with these activities in their own work, or be inspired to include art in math classes.

1. Introduction.

"My personal perspective is that it is desirable not to divorce truth and beauty; indeed, it is impossible. Trying to, limits the teacher and the students. Being open to beauty in mathematics stimulates love for it, and that, among other things, makes better mathematics." I wrote this years ago to explain my inclination to combine math and art; it resurfaced in a prospectus for my fall 2002 freshman seminar, *Mathematics and Esthetics—Science and Art in the Bay Area*. The original combination was a puppet show developed for a third grade class I taught through Project SEED [5]; the show elucidated the properties of a small group. In college, a second and grander puppet show demonstrated Galois theory to a modern algebra class. SEED's math enrichment philosophy is that novel material, usually presented socratically, can be satisfyingly grasped by most (they say all) students. This setting becomes a vehicle for establishing, reinforcing, or correcting standard math. I have used this approach in every grade 3–17. Peer teachers I trained through UC Berkeley delivered such curriculum to entire 5–10th grade classes for seven years.

Other purposes of my puppet adventures have been to lighten and enliven the classroom, to help students recognize that art is both a mode of discovery and a medium of expression in math, and to excite further explorations. These capitalize on the common fact that complex math may have simple yet nontrivial illustrative cases that all students can appreciate.[1] As in a good foreign language program, the student first gains a concrete immersive example—she has conversational fragments ahead of grammar.

Given generous and experienced readers, I shall render this material without the charm of the puppets or the support of lectures that the class audience gets. Not a happy compromise, but performances are evanescent, tutorials are lengthy. Readers may nonetheless find inspiration for bringing their own art and artistic insight to classroom settings, even if they don't normally teach mathematics. Mathematicians, bring art into your classrooms! Artists, bring yourselves there!

[1] For example, the Prisoners' Dilemma, a puzzling nonzero-sum two-person game, and Three-man Odd demonstrate Nash equilibria and afford wider appreciation of John Nash's Nobel-prize winning work. John Nash was my instructor in freshman honors calculus, probably the last course he taught—another instructor replaced him after the first month or so. For more on the Prisoner's Dilemma, see [2].

Figmentary reviews:

"If Shaw and Shakespeare can put a play in a play, why not a puppet show?"
"I applaud the way they talk about small groups; they cooperate rather than compete."
"—and ya gotta love the story—how the math and the myth work out together."

2. *The Four Alchemists.*

My first classroom puppet show was offered to the third-graders as a reward and as an application and extension of work they had completed with integer laws of multiplication and exponentiation. These and new laws would appear in the algebra underlying the story. The show portrayed the internal struggle for understanding among a guild of alchemists who transform matter. The Four Alchemists who represent (in dramatic and group-theoretic terms) this group are the three apprentices Typeman, Sizeman, and Fickle, and the master Nurd. Typeman changes gold to straw and vice-versa; Sizeman makes large objects small and small ones large; Fickle does both these actions at once, and Nurd does nothing. Their actions are depicted on their flag (Fig. 1). Figure 2 explains their group logo and individual flag symbols.

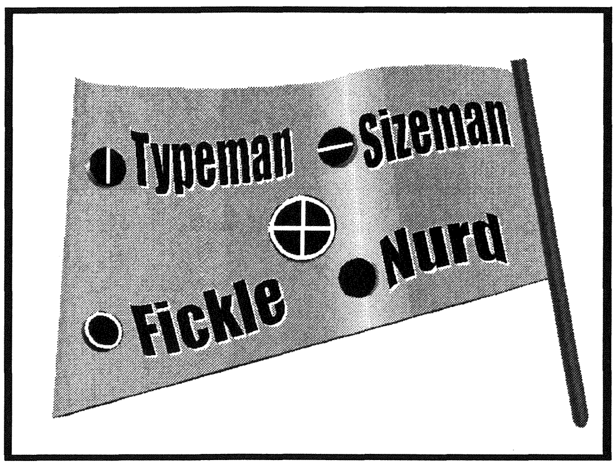

Figure 1. *Flag of the Four Alchemists.*

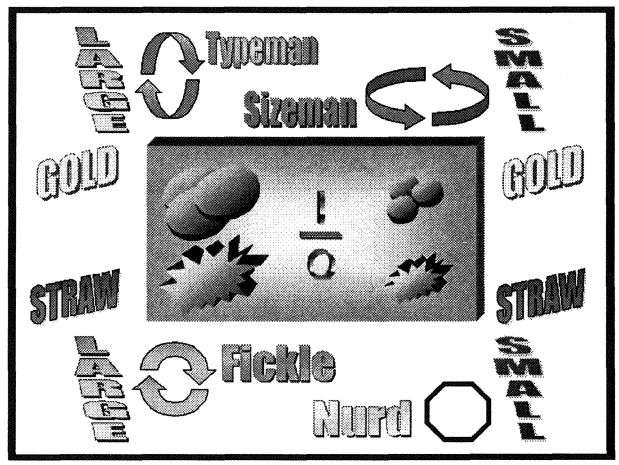

Figure 2. *Rectangle of activity of the Four Alchemists.*

*	N	T	S	F
N	N	T	S	F
T	T	N	F	S
S	S	F	N	T
F	F	S	T	N

Their activities commute and compose to form a group,[2] the Klein 4-group (or the Viergruppe, Vierergruppe, fours-group, action group, axial group, etc. [10, p. 27]). This group, designated as V_4 or more often V, is the direct sum of two two-element groups: $V = Z_2 \oplus Z_2$.[3] It is the only noncyclic four-element group and the smallest noncyclic group. It serves[4] as the symmetry group of the nonsquare rectangle (equiangular, but not equilateral) and also of the nonsquare rhombus

[2] A *group* is a set H of elements with an associative binary operation $\bullet$ on them that has an identity element 1 and a unary inverse operation $^{-1}$ so that for all a,b,c in H, $a\bullet(b\bullet c) = (a\bullet b)\bullet c$, $1\bullet a = a\bullet 1$, and $a\bullet a^{-1} = a^{-1}\bullet a = 1$. For example, the integers are a group under + and the positive rationals are a group under ×. Both of these groups are *commutative*: $a\bullet b = b\bullet a$ for all elements a,b. For more on groups and other algebraic structures, see, e.g., [4,10].

[3] Z_n is the finite additive group of congruence classes of integers mod n; that is, an element of Z_n is the set consisting of all the integers that have same remainder on division by n. The Z_n and integer group Z are called *cyclic*, because they are generated by a single element. When multiplication is included, these become *rings*.

[4] The same group may be identified in several ways by use. Groups are *isomorphic* if their operations yield the same multiplication tables on renaming the elements appropriately. But here it is easier just to say isomorphic symmetry groups are the same.

(equilateral, but not equiangular).[5] In addition, it is the dihedral group[6] D_2. It is elegantly simple, yet it turns up in numerous and diverse settings,[7] as we shall see.[8]

By the end of the show, the age of alchemy has mutated into better living through chemistry, romance is lost, the guild has become a union, the apprentices are journeymen now calling themselves Turn, Spin, and Flip, and Nurd has become Nothing. They have found that everyone is connected —because TS = F, SF = T, and FT = S—and each one of them is dispensable, but that means the others can cover his work during vacations and conferences. They also find that no one acts alone, for none of them can do all of the work of the other three. Well, almost ... . When they call Nothing a featherbedder because he does nothing, he points out that in fact that he does twice as much as each one of them, for TT = SS = FF = N, *and* the same work as all three working in a row, because TSF = TFS = ... = FST = N!

3. The briefest introduction to Galois theory.

We turn to *The Rootsellers*. This second puppet show was developed and performed for modern algebra classes to provide a final integration of the algebraic structures—groups, rings, fields, and lattices—that we had studied. Moreover, the puppet shows and other art demonstrations serve the students as models of popular arts-based term projects that I have required or suggested in many math and CS courses.

Instead of introducing the full abstract Galois theory, I chose one example of its application (a similar example can be found in [4, Ch. 11]). It illustrates the action of the Galois group of the quartic polynomial $p(x) = x^4 - 5x^2 + 6$ over the field[9] generated by the coefficients of p. This field is Q, the field of rational numbers, which is also its own minimal subfield, or prime field.[10]

The polynomial p reduces (factors) to $(x^2 - 2)(x^2 - 3)$, but these factors are irreducible over Q; the *roots* of p are the solutions of $p = 0$: the irrational real numbers $\pm\sqrt{2}$, $\pm\sqrt{3}$. The polynomial p reduces to linear factors—or splits—in the algebraic extension field $F = Q(\sqrt{2}, \sqrt{3})$ generated by these roots. The *Galois group G* of p is defined to be the group of field automorphisms[11] of the splitting field F that leave

[5] The symmetry groups we encounter here are subgroups of the group of rigid motions and reflections (isometries) of the 2-sphere; there are no infinitistic glide symmetries or infinite frieze or lattice groups. Note the distinction between symmetry groups and the *symmetric groups* S_n; the latter are all permutations of a set with n members, so S_n contains $n!$ elements. For $n > 3$, there is no geometric symmetry involving the entire group S_n.

Symmetry groups have standard designations in algebra, but they enjoy multiple names and notations in applications such as crystallography. Here are some equivalents for the groups encountered below (see [3]). V is also written as D_2 and 222 and is isomorphic to (i.e., has the same multiplication table as) C_{2v} ($mm2$) and C_{2h} ($2/m$). Z_2 is seen as C_2 (or 2), C_s (C_{1h} and $2/m$), and C_i (S_2 and -1). S_3 is commonly called D_3 (or 32) but also C_{3v} ($3m$). Z_3 (isomorphic to A_3) is C_3 (or 3) D_n usually keeps the name, although there may be alternatives. V_8 is written D_{2h} or V_h (also, mmm, $2/m\ 2/m\ 2/m$).

[6] The nth dihedral group D_n is the symmetry group of the regular n-gon; for $n = 2$, this degenerates to a "thick" line segment that has a two-fold rotation axis along its length, in particular.

[7] Klein encountered this group as the symmetry group of the nonregular right tetrahedron, which is the first (and degenerate) member of the sequence of right antiprisms, whose symmetry groups are the even dihedral groups D_{2n}. The nth antiprism has regular n-gons for top and bottom, rotated by a $1/2n$ turn, and joined up by $2n$ triangular faces, half pointing up and the other half down. When $n = 2$ this gives you V: the top and bottom n-gons degenerate to orthogonal skew segments, and only the four triangles remain (per Michael Kleber [6]).

[8] For applications beyond the framework of *The Rootsellers*, consult the Appendices—but there are even more.

[9] A *field* is a set K with two operations on it, called + and ×, and two elements 0, 1 such that K and + are a commutative group with identity 0, the nonzero elements of K and × are a commutative group with identity 1, and + and × are connected by distributivity: $(a+b)\times c = a\times c + b\times c$ for all elements a,b,c of K. The rational numbers, the real numbers, and the complex numbers are examples of fields. The ring Z is not, but Z_n is, just in case n is prime.

[10] Q has *characteristic* 0, for you cannot add 1 to itself repeatedly and ever obtain 0. Q is the unique prime field of characteristic 0.

[11] An automorphism of K is a one-to-one correspondence between K and itself that preserves the operations of addition and multiplication. Group and lattice automorphisms are defined similarly, using their own operations.

the coefficient field Q pointwise fixed (invariant).[12] It turns out that G is always some group of permutations on the roots of the polynomial, but not all permutations are in our G; for example, $\sqrt{2}$ cannot map to $\sqrt{3}$ for then 2 would go to 3, violating the fixing of Q. In fact, the Galois group of p is none other than the Alchemists' V!

Aside for the mathematically inclined: The heart of Galois theory is that it establishes a one-to-one correspondence between the subgroups of G and the fields lying between F and Q, by mapping each subgroup to the field of elements fixed by all the maps in the subgroup. The correspondence is a dual isomorphism between the lattice[13] of subgroups of G and the lattice of subfields of F. Moreover, the index of a subgroup is the degree of the invariant field over Q, and the order of the subgroup is the degree of F over that field. Our F is a formally real field and a subfield of the uniquely orderable field R of real numbers, which sits in the unorderable field C of complex numbers, the degree 2 algebraic closure of R.

Aside for algebraists: Because Q is char 0, F is separable; because V is abelian, all the intermediate fields are normal extensions. For a review of Galois theory, see [1].

4. *The Rootsellers.*

Shortly before this puppet show was created, I had regularly drunk a medicinal tea created from Chinese herbs, seeds, barks, and roots. I had also recently spent three weeks in China, including several days in the Huangshan (Yellow Mountain) area of Anhui province. This must account for the Chinese influence on my thinking when I meditated on a plot and a polynomial that would be appropriately simple to demonstrate Galois theory in a nontrivial way. I may not have started by choosing the Galois group G. When I did, it is possible that I did not immediately pick V as the culprit, but the coincidence was happy and the polynomial easy enough to construct. Although the two settings are distant in time and space, it is dramatically workable to include an abbreviated *Four Alchemists* as a scene in the larger show.

In the Chinoise setting of the new play, the four Rootsellers work just outside the boundary of the Garden of Kiu within the Vale of the dragon Long Riu Lain, all lying on the oceanic prairie of the Hai Si. Their names and their shading codes are given in Figure 3, and an overview of their fields in Figure 4.

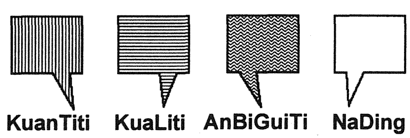

Figure 3. *Rootsellers and shading codes.*

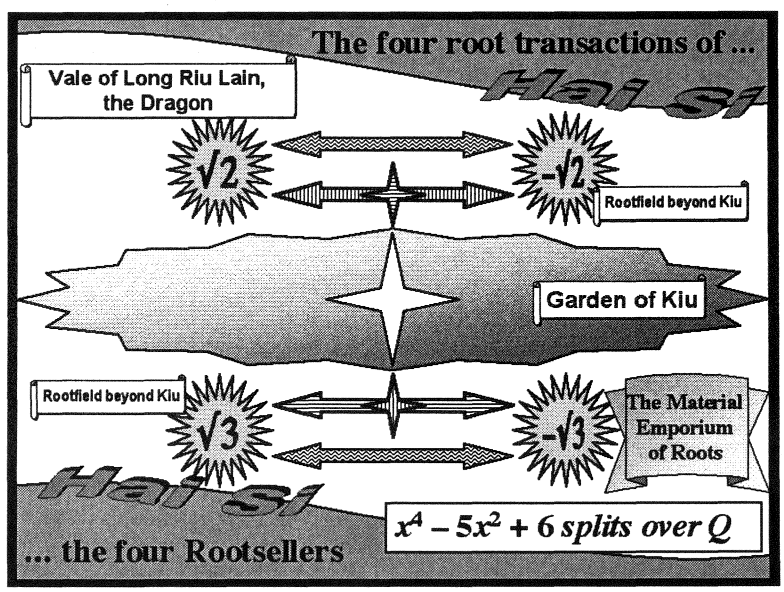

Figure 4. *Rootsellers' work and rootfields beyond Kiu.*

[12] Every automorphism of K will have this property automatically, because the coefficient field is also the prime field. Prime fields are invariant under all automorphisms—they admit only the trivial, identity automorphism.

[13] A *lattice* is a partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound. These provide two operations, called *join* and *meet*, with various useful properties. Somewhat misleadingly, these are often written as + and ·, especially in those lattices that are Boolean algebras.

In the past, the Rootsellers discussed their work purely in terms of the yin and yang energies of the roots Troo and Tree (as they call $\sqrt{2}$ and $\sqrt{3}$), which they cultivate, dry, and sell at market. KuanTiTi shifts Troo to -Troo and vice versa, KuaLiTi adjusts the balance of Tree and -Tree, and AnBiGuiTi, the Angel of Uncertainty, fiddles with both. NaDing, like Nurd, does nothing, but makes even more out of this as a positive contribution. These activities in the rootfields beyond Kiu are charted in Figure 4.

They are hard workers, and none are haughty, especially NaDing, but none thinks much about life far beyond Kiu, except NaDing.

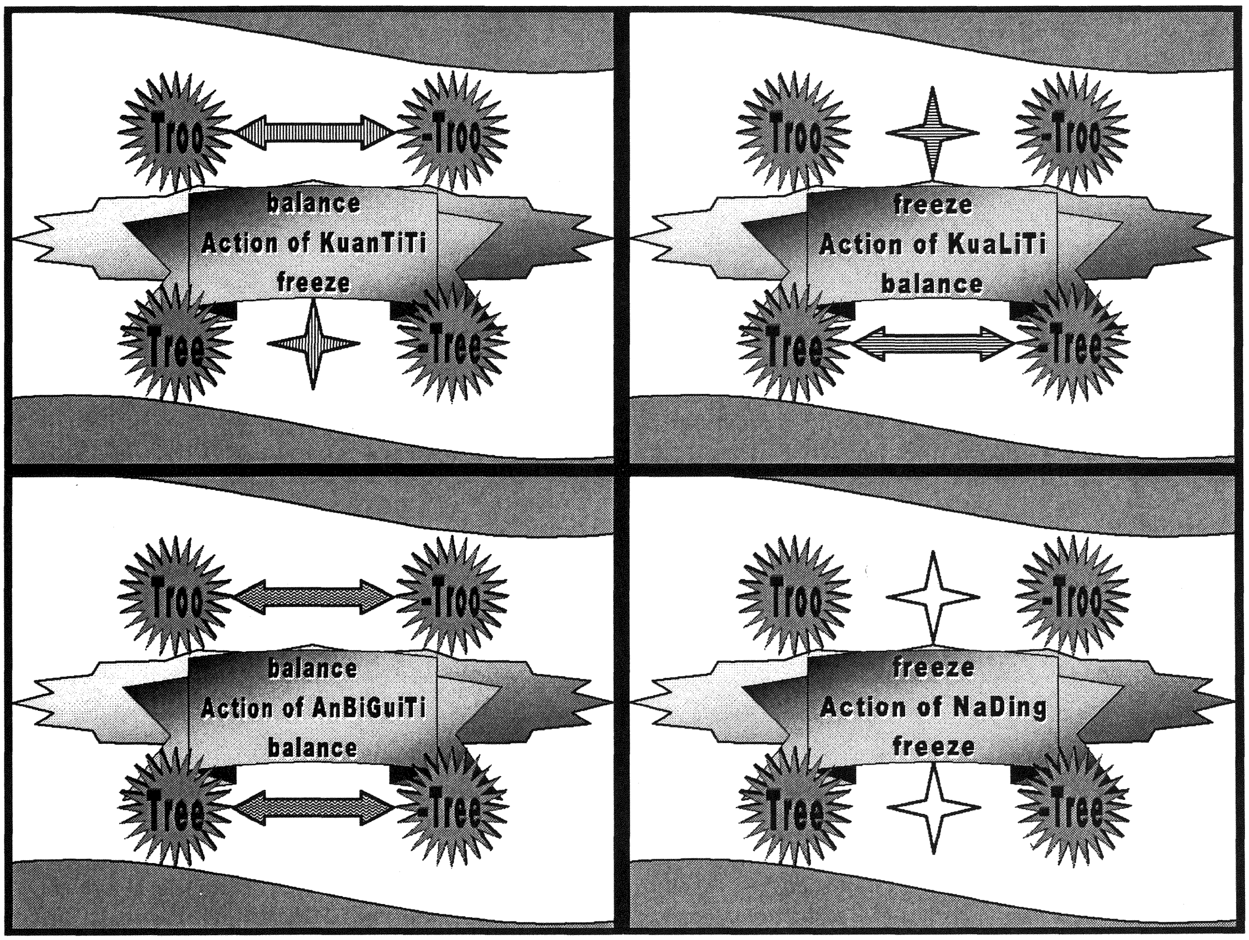

Figure 5. *The work of the individual Rootsellers.*

5. The world beyond the Garden of Kiu.

One day, when the time is ripe, NaDing organizes a retreat. In that quiet space, he suggests that rather than focusing on the roots they change, they should consider what they don't change, that is, leave fixed—or freeze. KuanTiTi fixes Tree, KuaLiTi freezes Troo, and NaDing freezes both. Their work is detailed in Figure 5. But they wonder what AnBiGuiTi fixes. Finally, NaDing reveals his enlightenment by pointing out that An freezes the product of Tree and Troo, which he calls Sax (for $\sqrt{6} = \sqrt{2} \times \sqrt{3} = -\sqrt{2} \times -\sqrt{3}$).[14] This is a "hidden" root: although it is in the rootfields beyond the Garden of Kiu, it is not one that they work with, and no one can even see it—except NaDing.

[14] Not a root of p, Sax still lies in Fa, or the splitting field F of p.

Then, much as the alchemists learned in the other show, the three working Rootsellers come to a deeper understanding of their actions and their interrelatedness (Fig. 6).[15] For example, any two of the three can do the work of the other two Rootsellers, but no one worker can, just like the Alchemists (anticipating their learning that V is the group of their actions). In other language, V is 2-generated three ways, but no element has order 4; that is, no element can generate the whole group: it is not cyclic.

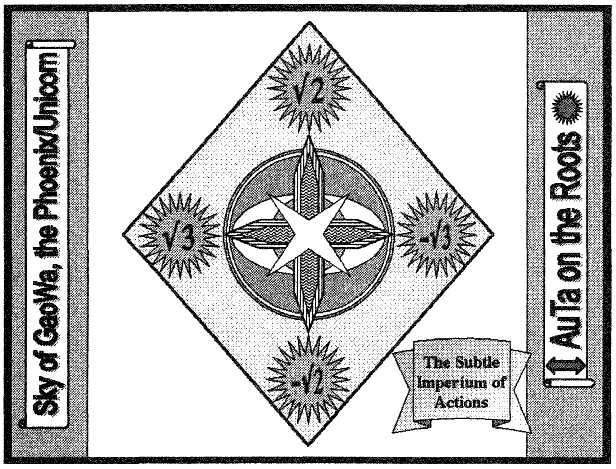

Figure 6. *The abstracted actions of the Rootsellers.*

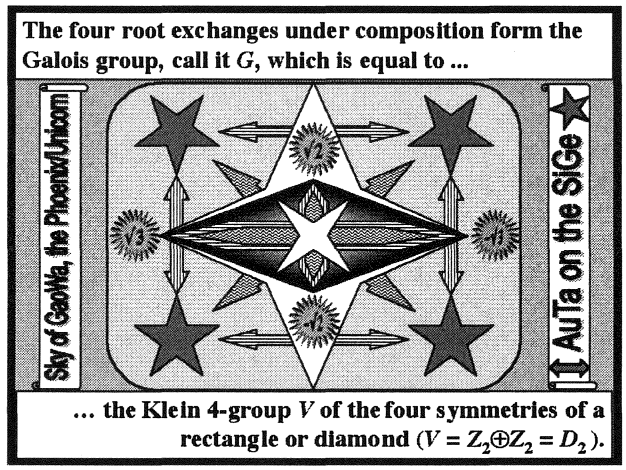

Figure 7. *Alchemists and Rootsellers act the same.*

With a rounded understanding of their own work, they now find they have joined a larger dance than they ever realized existed. Their journey of discovery shows them the correspondence between the phoenix and the dragon, empyrean alchemy and subterranean root energy, Heaven and Earth.

First, they see the identification of the symmetries of their actions in the diamond and the symmetries of the alchemists' actions in the rectangle (Fig. 7). Their abstracted actions form the Galois group G, which they now discover—with the help of some visiting Alchemists—is the well-known V.

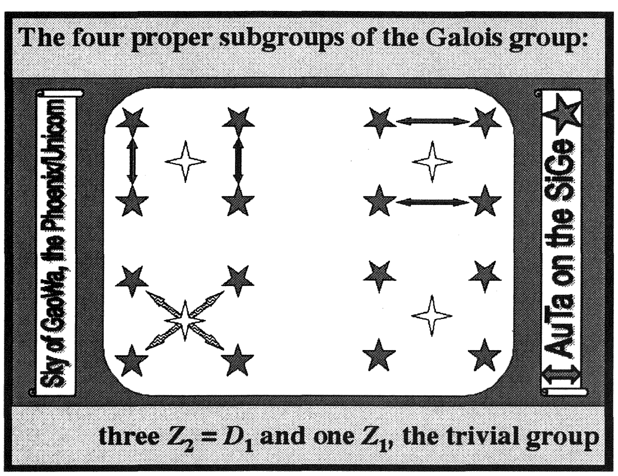

Figure 8. *Proper subgroups of the Galois group G.*

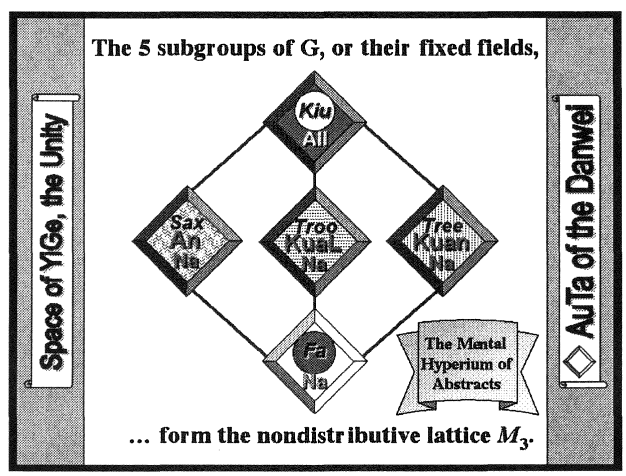

Figure 9. *Lattice of subgroups of the Galois group G.*

[15] In this and the other charts, fanciful labels suggest the original puppetry. One explanation: "avatar" is a common computing name for a representation of the user in a virtual world. "Avatar" becomes "AuTa," and the abstracted four working on the four vertices of the rectangle become the avatars acting on the four things or "Auta on the SiGe" (see Figure 7). Other avatars are apparent. For the Avatar, see [7].

Each field worker participates in a working subgroup he or she calls danwei. The four proper subgroups of V are shown in Figure 8. Note that NaDing is in every workgroup—for he is an agent, not of the state but of the static.

Their five danwei form the nondistributive modular lattice M_3 under the partial order of set inclusion (Fig. 9). NaDing's minimal workgroup is the unique, trivial, one-element group, $S_1 = Z_1 = D_0$.

From the vantage of their new level of understanding, the Rootsellers find that their subgroups leave invariant the subfields of F extending Q (that is, beyond the Garden of Kiu) generated by the roots they freeze, Troo, Tree, and Sax (Fig. 10). NaDing sings the name of the rootfield containing all: Fa.
The dot of the opposite color at top and bottom of Figure 9 and the root names suggest that when flipped upside down the lattice of subfields between F and Q results; upside down because bigger subgroups fix smaller fields. This double labeling is the heart of the Galois connection. Because M_3 is self-dual, i.e., isomorphic to the lattice obtained by reversing the direction of its partial order, these lattices are the same.

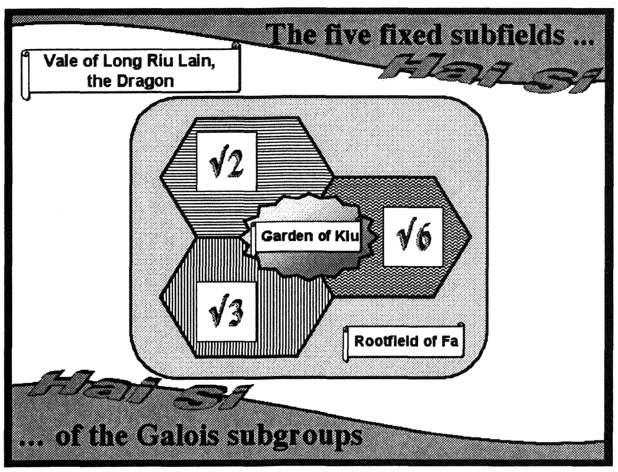

Figure 10. *The fixed subfields of the subgroups of G.*

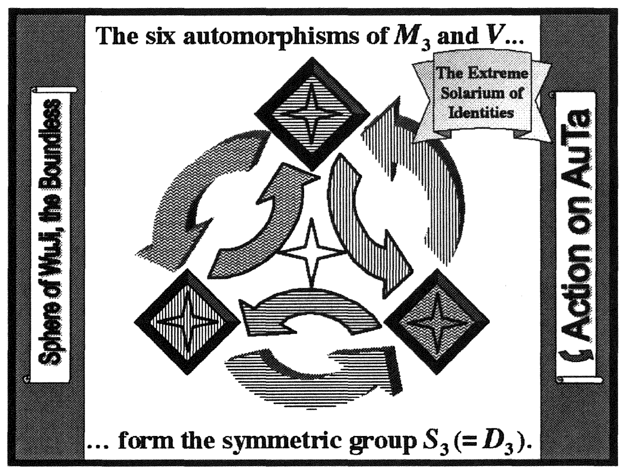

Figure 11. *The automorphism group of G itself.*

6. The Rootsellers, and their subgroups, are moved by S_3.

Ascending ever higher, they observe that they themselves in turn are acted on by the automorphism group of V, which is found to be the symmetric group S_3, all the permutations of 3 objects. S_3 can be represented as the group of symmetries of the equilateral triangle—the regular 3-gon; this means it is also D_3. In Figure 11, all six elements are depicted by the arrows, coded by style and shading.

The Rootsellers observe that permuting two-element subgroups is the same as swapping the people that generate them, so an automorphism of M_3 is the same as an automorphism of V (Fig. 12), and S_3 is also the automorphism group of this lattice.[16] Thus they discover that they are sympathetically affected in exactly the same ways their fields and workgroups are affected by these automorphic changes. They feel that they have entered into a profound harmony with nature and with the abstract.

NaDing draws one more lesson. Permutations are either even or odd, depending on whether they can be written as an even or odd number of transpositions (the parity is invariant under rewriting). The alternating subgroup A_n of S_n comprises half the elements—the even ones. The alternating group A_3 is also Z_3 and the only three-element subgroup of S_3, represented by the rotations of the equilateral triangle in multiples of 120° (Figs. 13, 14).

[16] Even more dizzying, the automorphisms of S_3 itself form $D_6 = S_3 \oplus Z_2$, the symmetries of the regular hexagon. The Z_2 part of D_6 acts on the two ±120° rotations in S_3, and S_3 partof D_6 acts on the three line reflections in S_3 .

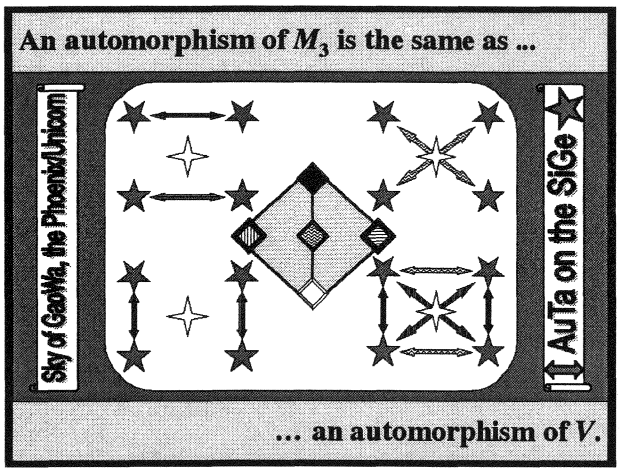

Figure 12. *Automorphisms of M_3 and V are the same.*

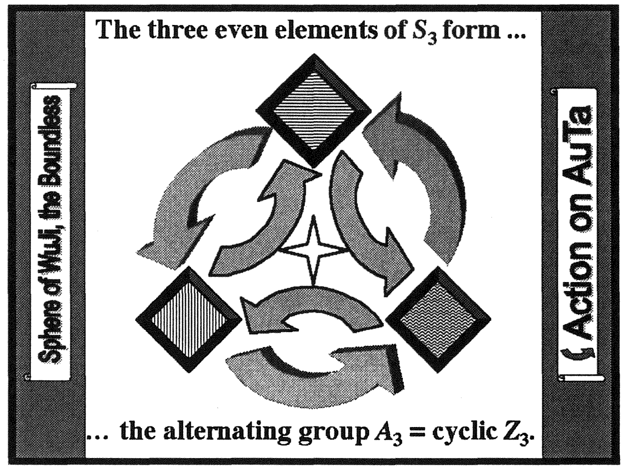

Figure 13. *The alternating subgroup.*

An element of A_3 cycles the shades of the subfields, but not the roots, for Q must remain pointwise fixed: the people change, but not the places (Fig. 14).[17]

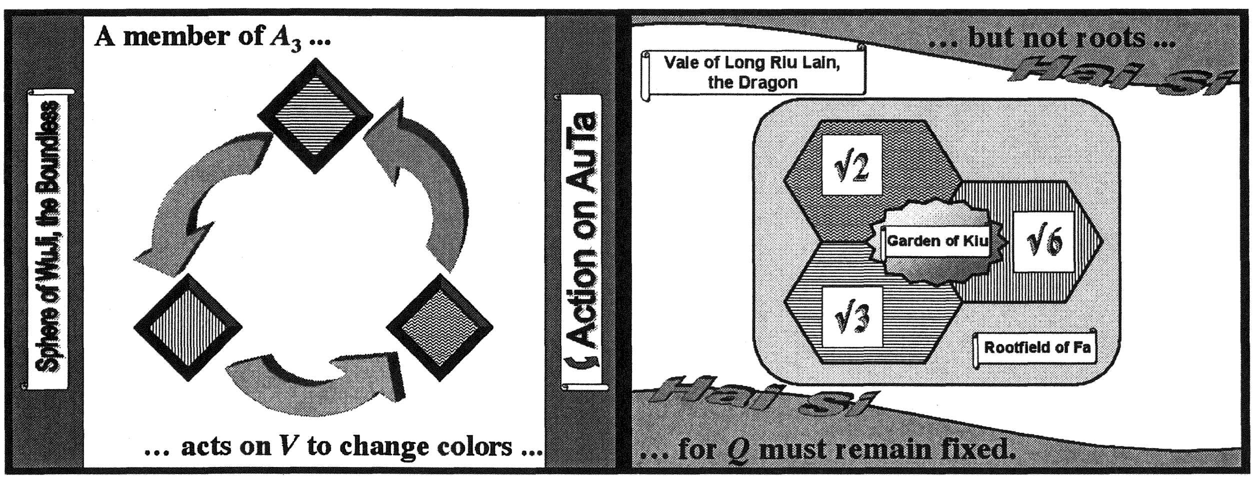

Figure 14. *An even automorphism acts on the subgroups(Fig.12) and the subfields in a similar manner.*

7. The Rootsellers come home, wiser for their inner journey.

The retreat ends on a pleasant note, namely fa; having explored the structure of the Rootfield of Fa, the quartet has discovered much about their quartic legacy. In Fa, they found their roots and factored linearly, became straightened. They have completed a new apprenticeship in understanding how by

[17] Any odd element of S_3 together with the identity element forms the subgroup $Z_2 = D_1$. Note that the regular 1-gon is then viewed as having a single edge with thickness and a crosswise orientation (an inside and outside), so it has the symmetry of the letter M. It is interesting that both V and S_3 have three two-element subgroups, and any two of these subgroups generate the whole group. But V and S_3 are quite distinct. In particular, S_3 is not commutative and, although larger, does not include V as a subgroup. V permutes the four vertices of a rectangle (or diamond), so it *is* a subgroup of the symmetric group S_4. V is the inner automorphism group of D_4 and normal in it; $D_4/V = Z_3$. To gild the lily, GF(4) is the only field with four elements; its multiplicative group is Z_3, and its additive group is V.

serving others, they in turn are served. All misgivings about the behavior of Nothing vanish in gratitude for the insight they have gained on their actions and their passions. Completing many circles and cycles, the Rootsellers will reenter the Garden of Kiu with renewed vigor, balanced energy, and inspired perspective, ready to take their own roots more regularly and less seriously.

8. Conclusion.

As you can see from the Figures, the puppets have morphed again, this time into a characterfree slide show. The PowerPoint illustrations shown here were originally intended as a treatment/storyboard for a 3D computer graphics animation, but instead have been used as stills for gifts and art pieces. Now the material has alchemically changed into a paper. That's enough gold for now.

The Australian poet Francis Brabazon said, "Poetry is Truth made charming," with the capital intended. Taken as a goal, not a judgment, we might apply the same in lowercase to endeavors to uncover the beauty of mathematics, especially for those who figure math is peripheral to their lives.

Three appendices offer material that can be used to extend a puppet show, a talk, or a paper.

9. Appendix A: The 2-torus.

Sinking back toward the Garden of Kiu, the Rootsellers are impressed by one final connection. They find that the alchemical rectangle becomes a torus T^2 under parallel edge identification (to see this, cut along the "straight" circles on the torus below). Their arrows of activity flow along some principal geodesics (the shading balloons indicate the association), illustrating that the fundamental homotopy group[18] $\pi(T^2)$ is $Z \oplus Z$ and $\pi(T^2, Z_2)$ is again none other than V.

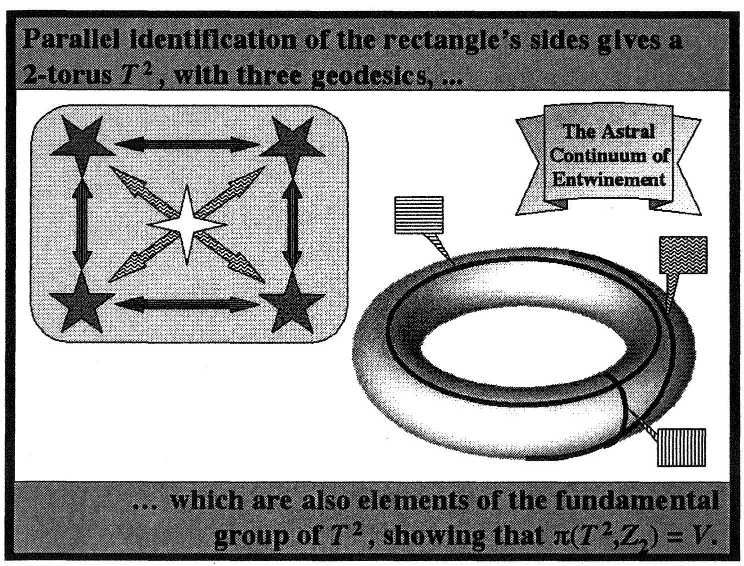

Figure 15. *Rolling the rectangle into a torus carries the avatars into geodesics.*

[18] The fundamental or first homotopy group is the group of equivalence classes of deformable oriented loops in the surface that begin and end at a fixed base point. The operation is a composition formed by running through one loop and then the next. When the same group is obtained no matter what the choice of base point is, then the base point is not mentioned. Here, one Z counts the number of times around the "short hole" (the donut hole), and the other, around the "long hole" (where the air in the innertube goes). Ignored are the infinite nonreentrant *skew lines*.

We could continue the exploration of topology and homotopy by considering the other well-known edge-identification manifolds: the Klein bottle (one edge-pair antiparallel) and the real projective plane (both edge-pairs antiparallel).

10. Appendix B: The Pascal-Sierpiński knots.[19]

Let $V = \{1,a,b,c\}$, so $aa = bb = cc = abc = 1$. Construct a version of Pascal's Triangle beginning with a top row consisting of a choice of two generators for the group.[20] As usual, the space beyond is filled with 1s and each element is the product of the two obliquely above it (Fig. 16). The omission of the first and last elements and the addition of the open loops at the corners are explained below.

If we make the two coordinate projections to Z_2 $\{1,a \to 0;\ b,c \to 1\}$ and $\{1,b \to 0;\ a,c \to 1\}$, each turns this into a copy of the usual Pascal's Triangle mod 2, offset horizontally (Fig. 17).

```
       ∩
      a b
     a c b
    a b a b
   a c c c b
  a b 1 1 a b
 a c b 1 a c b
⊂ b a b a b a ⊃
```

Figure 16. *V Pascal's triangle.*

```
        0 1                and          1 0
       0 1 1                           1 1 0
      0 1 0 1                         1 0 1 0
     0 1 1 1 1                       1 1 1 1 0
    0 1 0 0 0 1                     1 0 0 0 1 0
   0 1 1 0 0 1 1                   1 1 0 0 1 1 0
  0 1 0 1 0 1 0 1                 1 0 1 0 1 0 1 0
 1 1 1 1 1 1 1 1 1               1 1 1 1 1 1 1 1 0
```

Figure 17. *Mod 2 Pascal triangles from the V triangle.*

Continuing the mod 2 triangle produces an approximate likeness of the famed Sierpiński Gasket fractal (Fig. 18).

```
                               1
                              1 1
                             1 0 1
                            1 1 1 1
                           1 0 0 0 1
                          1 1 0 0 1 1
                         1 0 1 0 1 0 1
                        1 1 1 1 1 1 1 1
                       1 0 0 0 0 0 0 0 1
                      1 1 0 0 0 0 0 0 1 1
                     1 0 1 0 0 0 0 0 1 0 1
                    1 1 1 1 0 0 0 0 1 1 1 1
                   1 0 0 0 1 0 0 0 1 0 0 0 1
                  1 1 0 0 1 1 0 0 1 1 0 0 1 1
                 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1
                1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
               1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1
              1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1
             1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1
            1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1
           1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1
          1 1 0 0 1 1 0 0 0 0 0 0 0 0 0 0 1 1 0 0 1 1
         1 0 1 0 1 0 1 0 0 0 0 0 0 0 0 0 1 0 1 0 1 0 1
        1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1
       1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1
      1 1 0 0 0 0 0 0 1 1 0 0 0 0 0 0 1 1 0 0 0 0 0 0 1 1
     1 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 1
    1 1 1 1 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1 0 0 0 0 1 1 1 1
   1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1 0 0 0 1
  1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 1 1
 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1 0 1
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
```

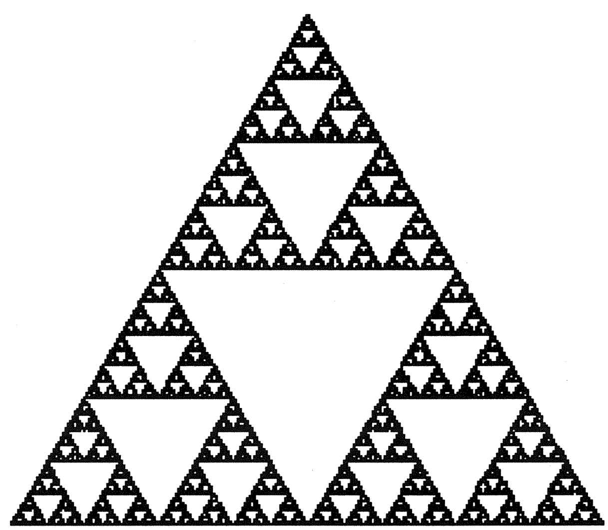

Figure 18. *The mod 2 Pascal triangle converges in a certain sense to the Sierpiński triangular gasket.*

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[19] I am indebted to Michael Kleber of Brandeis University for this material and most of its exposition [6]. He also points out that V is the group of eight quaternion units modulo ± 1.

[20] The choice of generators specifies the isomorphism from V onto $Z_2 \oplus Z_2$: a goes to <1,0>, and b to <0,1>.

Referring back to the partial V Pascal triangle in Figure 16, with an open loop at each corner, we obtain a prescription for the oriented crossings (1 means no crossing) of the third knot in the sequence[21] in Figure 19 elaborated from the leftmost trefoil knot. The sequence converges in an obvious geometric sense to the Sierpiński triangle.

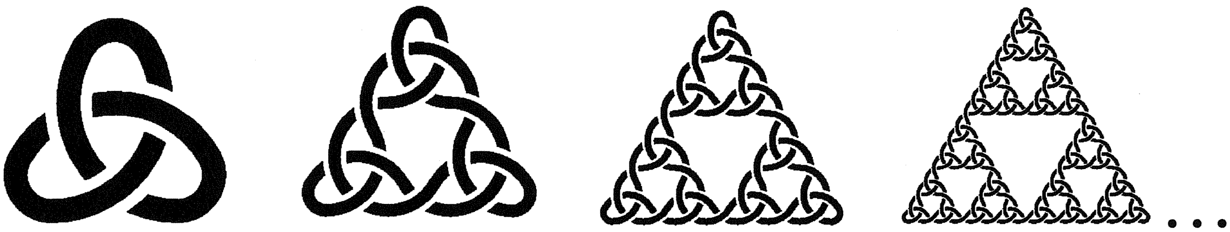

Figure 19. *Pascal-Sierpiński knot sequence.*

11. Appendix C: A generalization.

If we go to three dimensions, we can extend a lot of the Rootsellers' findings to the symmetries of a square-free rectangular solid R (general rectangular parallelepiped). The symmetry group of R is a group with 7 idempotent nonidentity elements: the three from V that, say, represent orthogonal two-fold axes of rotation, plus another three from a copy of V that represent orthogonal mirror planes, and a seventh that is point-reflection, or inversion, in the center of the solid.[22] This eight-element group (call it V_8—humorous, but not quite accurate) is $V_8 = V \oplus Z_2 = Z_2 \oplus Z_2 \oplus Z_2 = \{1,a,b,c,d,e,f,g\}$. While V is generated by three ($= C(3,2)$) different sets of two generators, V_8 is generated by $C(7,2)*4/3 = 28$ different generating sets. To count this, make a free choice of an unordered pair for the first two generators. There are four candidates outside the subgroup the two generate that can serve as the third generator. Divide by 3 to count correctly the unordered set made by adding this third element.

Let $<a,b,c>$ be some choice of an ordered triple of generators. Then Figure 20 can serve as a multiplication table; there are many tables to choose from, and some are not only diagonally symmetric (guaranteed by the commutativity of V_8), but also simple and rhythmic. This table may seem to have unmotivated complexity, but it gives a prettier picture for the lattice[23] of subgroups (Fig 21).

•	1	*a*	*b*	*c*	*d*	*e*	*f*	*g*
1	1	*a*	*b*	*c*	*d*	*e*	*f*	*g*
a	*a*	1	*f*	*d*	*c*	*g*	*b*	*e*
b	*b*	*f*	1	*g*	*e*	*d*	*a*	*c*
c	*c*	*d*	*g*	1	*a*	*f*	*e*	*b*
d	*d*	*c*	*e*	*a*	1	*b*	*g*	*f*
e	*e*	*g*	*d*	*f*	*b*	1	*c*	*a*
f	*f*	*b*	*a*	*e*	*g*	*c*	1	*d*
g	*g*	*e*	*c*	*b*	*f*	*a*	*d*	1

Figure 20. *Table for* V_8

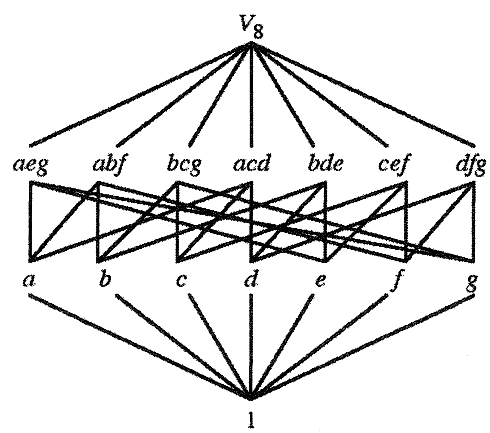

Figure 21. *Subgroup lattice for* V_8.

[21] These knots were drawn using Adobe PostScript code written by Kleber [6]. The printer actually calculates the V Pascal triangle to draw them. For another use of the V Pascal triangle, see [9].

[22] The last four are also fourth-dimensional rotations, just as T and F in V are mirror lines or 3D rotations.

[23] Thanks to JB Nation [8] for pointing out the table ordering in Figure 20, and the identification of the self-dual lattice in Figure 21 with the finite projective plane of order 2 (the top layer is the lines, and the bottom, the points).

There are altogether $C(7,3) = 35$ three-element subsets, 28 are generating sets for V_8, and the other 7 sets[24] of three elements are each the nonidentity elements in a subgroup with four elements. Knowing two of these elements in a given subgroup uniquely determines the third element: it is the product of the two. So there are seven subgroups of order 4, all copies of V. There are also 7 subgroups of order two, each generated by one of the nonidentity elements. This gives the lattice of subgroups listed by generators in Figure 21. Because M_3 embeds in it, it is also not distributive, although it is modular.[25] Because the subgroups are all normal, these self-dual lattices are also the congruence lattices of V and V_8.

The automorphisms of V_8 are specified by taking our particular ordered generating set $<a,b,c>$ (order counts!) and mapping it onto another, or itself. There are 28 targets, each orderable in 3! ways, so the total number of automorphisms is 168. The subgroup lattice has the same number of automorphisms —and in fact the same automorphism group of index 30 in S_7.[26]

We can get a polynomial yielding this Galois group by taking an additional square root of a prime, say $\sqrt{5}$ (Fave), and forming the polynomial

$$q(x) = (x^2 - 2)(x^2 - 3)(x^2 - 5) = p(x)\cdot(x^2 - 5) = x^6 - 10x^4 + 31x^2 - 30;$$

and noting that the degree of q does not divide the degree of the splitting field over Q, because the exponents add, but the field degrees multiply and group orders and indexes multiply.

The observed patterns (and more) extend naturally to higher dimensions—squarefree tesseracts and beyond—but the simplicity would be stretched and the span of Bridges exceeded.

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[8] J. B. Nation, personal communication.

[9] K. M. Shannon and M. J. Bardzell, *The PascGalois Project: Visualizing Abstract Algebra*, MAA Focus, March 2002, pp. 4-5. Also see: *PascGalois Triangles & Hexagons and other Group-related Cellular Automata*, http://faculty.salisbury.edu/~kmshannon/pascal/ (consulted 3/16/02).

[10] B. L. van der Waerden, Modern Algebra. New York: Frederick Ungar. 1953.

[24] This can also be computed as $7 = C(7,2)/3$, the number of unordered pairs cut to a third because three different pairs generate each order 4 subgroup (a copy of V).

[25] The subgroup lattice of a group is always modular: N_5, the smallest nonmodular lattice, does not embed in it.

[26] Automorphism groups of a group and its subgroup lattice generally differ. For example, for p prime, $\mathrm{Aut}(Z_p) = Z_{p-1}$, but $\mathrm{Lat}(Z_p)$ is the rigid, two-element lattice, and so $\mathrm{Aut}(\mathrm{Lat}(Z_p))$ is trivial.