Bridges
Mathematical Connections
in Art, Music, and Science

A Musical Suite Based on the Platonic Solids

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Abstract

We describe here a suite for unaccompanied violin which is based on the five Platonic solids. This ties together our own career in computer science with our lifelong interest in the composition of music, including works for symphony orchestra and for theatrical band and orchestra with chorus and soloists. A consequence of the design of the suite is that every measure in it is played twice, sometimes in immediate succession but more often at a later point in the same movement. The sheet music to the suite is available from the author.

Introduction

We describe here a suite for unaccompanied violin whose construction is based on that of the five Platonic solids: the cube, the tetrahedron, the octahedron, the dodecahedron, and the icosahedron. The suite has five movements, one for each Platonic solid. The aims in constructing the suite were to provide a pleasing musical suite in the style of those written for unaccompanied violin in the eighteenth and nineteenth centuries, while at the same time using as many properties of the Platonic solids as possible, and relating them to the music. The sheet music to the suite is available from the author.

A general theme throughout the suite is concerned with an analogy between the twelve notes of the musical scale, the twelve vertices of the icosahedron, the twelve faces of the dodecahedron, the twelve edges of the octahedron, and the twelve edges of the cube. Consider, for example, the third movement, based on the octahedron. This has twelve edges, and we assign to each edge a measure of the piece. Each of these measures has a starting note and an ending note, so there are 12 starting notes, all different, and also 12 ending notes, all different. The tetrahedron has six edges, but the analogy with the number 12 still applies, because there are six measures, each of which has a starting note and an ending note, and all of these 12 notes are distinct.

The use of twelve different notes in several places, in each of these pieces, should not be confused with the construction of twelve-tone music. In twelve-tone music, there is no attempt to resemble the sound of eighteenth-century or nineteenth-century music. On the other hand, eighteenth-century music usually does not use all twelve tones (Bach's Well-Tempered Clavier being a famous exception, since it contains one movement in each of the major and minor keys). Our music is more reminiscent of the style of Cesar Franck, particularly the D major symphony, in which a wide variety of chords is used.

The first movement of the suite is based on the icosahedron; the second, on the cube; and the third, on the octahedron. The last movement is based on the dodecahedron, and it has a short introduction, based on the tetrahedron. This introduction is actually a separate movement, although to the listener's ear this may not appear to be the case; the piece may well sound like a standard four-movement suite with an introduction to the last movement.

The time signature of each movement is based on the number of edges in a face of the corresponding solid. Thus the cube has four edges to a face, and the second movement, based on the cube, is in 4/4 time.

The dodecahedron has five edges to a face, and so the last movement, based on the dodecahedron, is in 5/4 time. Each of the other Platonic solids has triangular faces, and so each of the other three movements in the suite is in 3/4 (or, in the case of the interlude, 3/2) time.

The suite is entitled Platonic Suite In A Minor (A for Athens, Plato's home). Following a well-known eighteenth-century practice, although this is called a suite in A minor, only the first movement is actually in A minor; the others, respectively, are in F major, C major, C major (the interlude), and A major.

Each movement of the suite is divided into as many subparts as there are faces of the corresponding Platonic solid. Each face, in turn, is divided up into the edges of that face. As mentioned above, each edge corresponds to a measure of the piece. When playing a subpart corresponding to a particular face, the measures corresponding to the edges of that face are played in order, going aound the face, and then this continues for all the other faces.

Each edge of a solid, of course, is part of two different faces, and it follows from what has just been said that, in each movement, each measure is played twice and exactly twice. In the second, fourth (interlude), and fifth movements, as many measures as possible are played consecutively twice. In the other two movements, however, this is not the case. Each measure, then, has had to be constructed in such a way that it will make musical sense in two different contexts, that is, with two different measures preceding and following.

We recall that all the Platonic solids, except for the cube, have names reflecting their respective numbers of faces. Thus there are the tetrahedron, with four faces (tetra = 4); the octahedron, with eight faces (oct = 8); the dodecahedron, with 12 faces (dodeca = 12); and the icosahedron, with 20 faces (icosa = 20). Descriptions of the individual movements now follow.

First Movement (A Minor) — The Icosahedron

The icosahedron has 20 faces, each of which is a triangle, with three edges. It therefore has $(20 \cdot 3)/2 = 30$ edges; it also has 12 vertices. Accordingly, the first movement is divided into 20 subparts, each of which has 3 measures, for a total of 60 measures, or 30 distinct measures, each of which is played twice. It is in 3/4 time, since a triangle has three edges.

Figure 1 represents an icosahedron in two views, one looking down at the top and the other looking in at the inside of the bottom. It is assumed that the icosahedron is balanced on one of its vertices, and we are looking down on the top of it. Unlike the other Platonic solids, the icosahedron has its vertices labelled here, as well as its edges. We denote the vertices by V1 through V12; and, to each vertex, there corresponds a different note of the musical scale, as follows:

VERTEX	V1	V2	V3	V4	V5	V6	V7	V8	V9	V10	V11	V12
NOTE	A	C	Eb	G	C#	F	G#	Bb	B	D	F#	E

There are five edges radiating from vertex V1, which we denote as edges A, B, C, D, and E. These lead to vertices V2, V3, V4, V5, and V6, which form a regular pentagon, the edges of which are denoted by F, G, H, J, and K. This is on the top of the icosahedron, and the same happens at the bottom: there are five edges rating from vertex V12, which we denote as edges R, S, T, U, and V, and these lead to vertices V7, V8, V9, V10, and V11, which form a regular pentagon, the edges of which are L, M, N, P, and Q. There are then ten more edges connecting the top to the bottom of the icosahedron, which we denote by 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

Digression — A Mathematical Impossibility

Of all the Platonic solids, the icosahedron is the only one for which we have associated unique notes with vertices, rather than with faces or edges. As we go around each triangle, in each subpart of the first movement, it might be supposed that the initial notes corresponding to its three vertices would always be played. Unfortunately, this is a mathematical impossibility if each measure is to be played exactly twice. This arises from the fact that five (an odd number of) triangles come together at each vertex.

Suppose, for example, that we go from vertex V1 to V2 to V3 around a triangle. The edge from V1 to V2 would then start with V1, but the edge from V3 to V1 would start with V3, not with V1. Therefore, we could not go from vertex V1 to V3 to V4 around the next triangle. If we did, then the edge from V1 to V3 would start on V1's note, whereas we have just seen that it starts with V3's note, which is different from that of V1. It follows that, when going around the next triangle, we must go from V1 to V4 to V3, not from V1 to V3 to V4. By the same logic, we must go around from V1 to V4 to V5 (not V1 to V5 to V4), and then from V1 to V6 to V5 (not V1 to V5 to V6). When it comes to the triangle containing V1, V6, and V2, we are now stuck; either way is wrong.

Another way to say this is that an icosahedron cannot be considered as a directed graph with a directed path around each of its 20 triangles. If it were, then at each vertex Z, the edges involving Z, considered in clockwise order, must come alternately into Z and out of Z. This, however, is impossible because such an alternation scheme would require that the total degree at each vertex be even.

Because of this, we are using a separate scheme for the top and bottom vertices, V1 and V12. Every measure involving V1 starts with V1's note (A), and every measure involving V12 starts with V12's note (E). This also makes esthetic sense if the measures involving V1 are all at the beginning, and the measures involving V12 are all at the end. In this way, ten measures (five, each played twice) are near the beginning of the movement, and each one starts on the tonic (A); also, ten measures (five, each played twice) are near the end of the movement, and each one starts on the dominant (E).

This accounts for ten of the 20 triangles in the icosahedron. We now can, and do, go aound each of the remaining 20 triangles playing three different vertex-related starting notes each time. The directions along the top pentagon are from V2 to V3 to V4 to V5 to V6 to V2; those along the bottom pentagon are from V7 to V8 to V9 to V10 to V11 to V7.

Design Of The First Movement (Continued)

Each edge, then, has a starting vertex and an ending vertex. The measure corresponding to each edge starts with the note associated with its starting vertex. There is no special note on which a measure ends. The 30 edges, then, each with its starting and ending vertex and starting note (as specified above), are:

EDGE	A	B	C	D	E	F	G	H	J	K
STARTING VERTEX	V1	V1	V1	V1	V1	V6	V2	V5	V3	V4
ENDING VERTEX	V2	V6	V3	V5	V4	V2	V3	V6	V4	V5
STARTING NOTE	A	A	A	A	A	F	C	C#	Eb	G

EDGE	0	1	2	3	4	5	6	7	8	9
STARTING VERTEX	V7	V3	V8	V4	V9	V5	V10	V6	V11	V2
ENDING VERTEX	V2	V7	V3	V8	V4	V9	V5	V10	V6	V11
STARTING NOTE	C	G#	Eb	Bb	G	B	C#	D	F	F#

EDGE	L	M	N	P	Q	R	S	T	U	V
STARTING VERTEX	V8	V9	V7	V10	V11	V12	V12	V12	V12	V12
ENDING VERTEX	V9	V10	V8	V11	V7	V9	V8	V10	V7	V11
STARTING NOTE	Bb	B	G#	D	F#	E	E	E	E	E

The subparts of the first movement, corresponding to the 20 triangular faces of the icosahe-dron, are:

EDGES	STARTING VERTICES	STARTING NOTES	EDGES	STARTING VERTICES	STARTING NOTES
A, C, G	V1, V2, V3	A, A, C	M, 6, 5	V9, V10, V5	B, D, C#
C, E, J	V1, V3, V4	A, A, Eb	K, 5, 4	V4, V5, V9	G, C#, B
E, D, K	V1, V4, V5	A, A, G	L, 4, 3	V8, V9, V4	Bb, B, G
D, B, H	V1, V5, V6	A, A, C#	J, 3, 2	V3, V4, V8	Eb, G, Bb
B, A, F	V1, V6, V2	A, A, F	N, 2, 1	V7, V8, V3	G#, Bb, Eb
G, 1, 0	V2, V3, V7	C, Eb, G#	P, V, T	V10, V11, V12	D, E, E
Q, 0, 9	V11, V7, V2	F#, G#, C	M, T, R	V9, V10, V12	B, E, E
F, 9, 8	V6, V2, V11	F, C, F#	L, R, S	V8, V9, V12	Bb, E, E
P, 8, 7	V10, V11, V6	D, F#, F	N, S, U	V7, V8, V12	G#, E, E
H, 7, 6	V5, V6, V10	C#, F, D	Q, U, V	V11, V7, V12	F#, E, E

Second Movement (F Major) — The Cube

The cube has six faces, each of which is a square, with four edges. It therefore has $(6\cdot4)/2 = 12$ edges; it also has eight vertices. Accordingly, the second movement is divided into six subparts, each of which has four measures, for a total of 24 measures, or 12 distinct measures, each of which is played twice. It is in 4/4 time, since a square has four edges.

Figure 2 represents a cube. Its top is ABCD; the edges from the top to the base are E, F, G, and H; and its base is JKLM. The measure corresponding to each edge starts on a unique note of the scale, and also ends on a unique note of the scale. The subparts are:

A-C-D-B B-E-K-G G-D-H-M M-K-J-L L-H-C-F F-J-E-A

The starting and ending notes of each of the 12 measures are:

MEASURE	A	B	C	D	E	F	G	H	J	K	L	M
STARTING NOTE	A	C	E	Bb	G	F#	D	F	Eb	C#	B	G#
ENDING NOTE	F	F#	C	C#	Bb	G#	A	D	G	E	Eb	B

Third Movement (C Major) — The Octahedron

The octahedron has eight faces, each of which is a triangle, with three edges. It therefore has $(8\cdot3)/2 = 12$ edges; it also has six vertices. Accordingly, the third movement is divided into eight subparts, each of which has three measures, for a total of 24 measures, or 12 distinct measures, each of which is played twice. It is in 3/4 time, since a triangle has three edges.

Figure 3 represents an octahedron, which, as we recall, looks like an ordinary rectangular pyramid set atop its reflection. The edges from the top of the pyramid to its base are A, B, C, and D; the base is EFGH; and the edges from the base to the bottom vertex (at the bottom of the reflection) are J, K, L, and M. As with the cube, the measure corresponding to each edge starts on a unique note of the scale, and also ends on a unique note of the scale. The subparts are:

A-B-E A-C-F B-D-G C-D-H E-J-K F-J-L G-K-M H-L-M

The starting amd ending notes of each of the 12 measures are:

MEASURE	A	B	C	D	E	F	G	H	J	K	L	M
STARTING NOTE	E	A	Ab	F#	F	Eb	D	Bb	B	C#	C	G
ENDING NOTE	G	E	F	C#	D	Ab	A	Eb	F#	B	Bb	C

Fourth Movement (C Major) — The Tetrahedron

The tetrahedron has four faces, each of which is a triangle, with three edges. It therefore has $(4 \cdot 3)/2 = 6$ edges; it also has four vertices. Accordingly, the fourth movement is divided into four subparts, each of which has three measures, for a total of 12 measures, or six distinct measures, each of which is played twice. A triangle has three edges, and this movement is in 3/2 time.

Figure 4 represents a tetrahedron, which is of course a triangular pyramid. The edges from the top to the base of the pyramid are designated as A, B, and C, and the base of the pyramid is DEF. Each of the twelve notes of the musical scale is either the starting note or the ending note of one of the six measures corresponding to the six edges of the pyramid. The subparts are:

A-D-B B-F-C C-A-E E-D-F

The starting and ending notes of each of the six measures are:

MEASURE	A	B	C	D	E	F
STARTING NOTE	C	A	G#	F#	F	D
ENDING NOTE	G	D#	Bb	B	Db	E

Fifth Movement (A Major) — The Dodecahedron

The dodecahedron has 12 faces, each of which is a pentagon, with five edges. It therefore has $(12 \cdot 5)/2 = 30$ edges; it also has 20 vertices. Accordingly, the fifth movement is divided into 12 subparts, each of which has five measures, for a total of 60 measures, or 30 distinct measures, each of which is played twice. It is in 5/4 time, since a pentagon has five edges.

Figure 5 represents a dodecahedron. As with the icosahedron, one of these views is looking down at the top, while the other is looking in at the inside of the bottom. The dodecahedron is assumed to be resting on its base, and its top is a level surface. The edges A, B, C, D, and E are those of the pentagon at the top, from each of whose vertices there extend further edges F, G, H, J, and K.

Suppose now that we think of the dodecahedron as separated into two equal parts. The first of these consists of the pentagon at the top, together with the five pentagons extending downward from it, for a total of six, as illustrated in Figure 5(**a**). The other is similar, involving the bottom pentagon and the five pentagons extending upward from it, as illustrated in Figure 5(**b**).

The bottom edge of the top piece of the dodecahedron is a serrated ten-sided figure in three dimensions, whose edges are designated 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. The top edge of the bottom piece is another serrated ten-sided figure, and these two serrated figures are now fit together, but with a half turn, so that the bottom face is rotated 180° with respect to the top face. The edges R, S, T, U, and V are those of the pentagon at the bottom, from each of whose vertices there extend further edges L, M, N, P, and Q.

The dodecahedron has twelve faces, and to each face there corresponds a unique note of the scale. Each subpart starts on that note and also ends on that note. The subparts are as follows, each with the note on which it starts and ends:

ABDEC (E)	0FAG1 (F#)	1QUN2 (D)	2GCJ3 (B)	3NSL4 (F)	4JEK5 (G)
5LRM6 (Eb)	6KDH7 (C)	7MTP8 (Ab)	8HBF9 (Db)	9PVQ0 (Bb)	TRSUV (A)

The starting amd ending notes of each of the 30 measures are:

MEASURE	A	B	C	D	E	F	G	H	J	K
STARTING NOTE	E	G	G#	B	A	B	G	F	Eb	D
ENDING NOTE	F#	C	E	G#	E	G	B	C	G	C

MEASURE	L	M	N	P	Q	R	S	T	U	V
STARTING NOTE	C	Eb	C	F#	G	A	E	A	B	C#
ENDING NOTE	E	G	G	A#	D	C#	G	C#	E	A

MEASURE	0	1	2	3	4	5	6	7	8	9
STARTING NOTE	F#	D	B	F	G	Eb	C	G#	C#	A#
ENDING NOTE	A#	F#	D	B	F	G	Eb	C	G#	C#

To retain the 5/4 time in the mind of the listener, there are two staccato eighth notes at the end of each measure in the movement, with the rest of each measure being played legato.

Conclusion

We have successfully demonstrated the possibility of constructing music with strong connections to the Platonic solids, while at the same time making it pleasing to the ear. The process of constructing the suite, as described in the paper, was tedious at times, but never involved any real difficulties. Only elementary mathematical and musical ideas were used. The author would therefore like to encourage others to write similar music based on other mathematical constructions.

Our Platonic Suite is an example of the kind of musical work which would have been far more difficult to write even twenty-five years ago, before musical notation processing systems had been developed. In particular, the usual notions of cutting and pasting, applied to musical notation, made it not only easier, but far less error-prone, to include each measure exactly twice. Among musical notation systems, the one we use is the one that has been in existence for the longest time; it is called Composer's Mosaic, from Mark Of The Unicorn, Inc.

FIGURE 1(a) FIGURE 1(b)

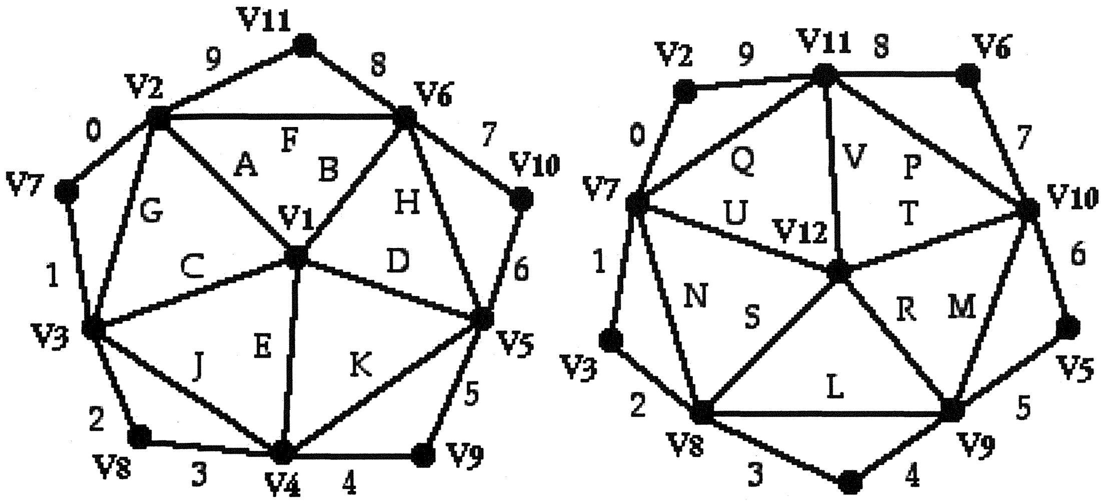

FIGURE 1. The icosahedron.

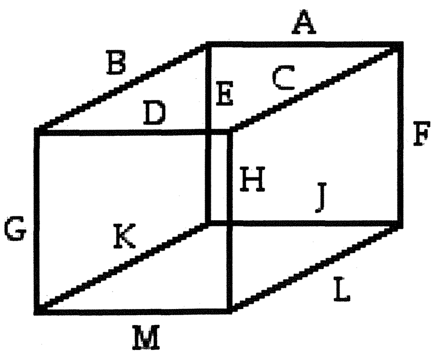

FIGURE 2. The cube.

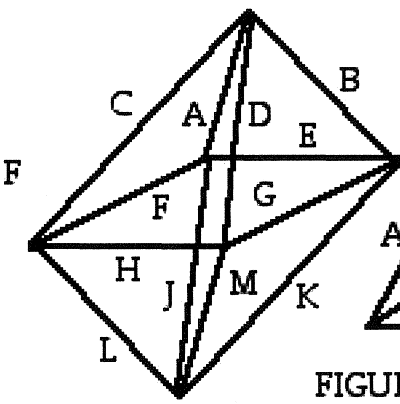

FIGURE 3. The octahedron.

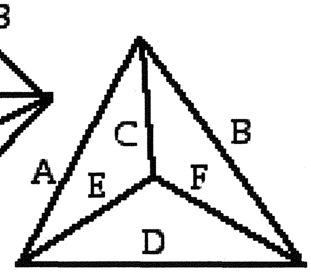

FIGURE 4. The tetrahedron.

FIGURE 5(a) FIGURE 5(b)

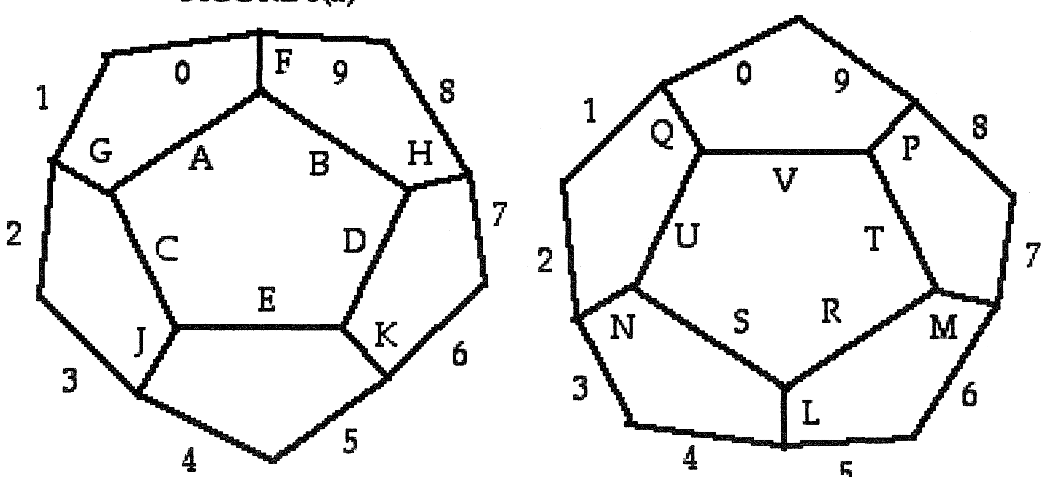

FIGURE 5. The dodecahedron.