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THE POETRY OF INFINITY

Exploring Mathematical Concepts as Metaphor In the Works of Jorge Luis Borges

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Abstract

Many of the short stories of Jorge Luis Borges have both specific mathematical objects and abstract mathematical concepts embedded in their very structure. The most prevalent of these is the concept of Infinity, which Borges examines not only as a philosophical idea, as many have done before him, but also as a precisely defined mathematical object. The mathematical properties of Infinity are incorporated into his stories as metaphors, lending a strange and magical feel to them. In this paper we examine four of his most famous stories and the mathematics they contain.

> "One concept corrupts and confuses the others. I am not speaking of Evil whose sphere is ethics; I am speaking of the infinite."
>
> -----------Jorge Luis Borges

Jorge Luis Borges

Jorge Luis Borges is one of the most important and influential of Latin American writers. He was born in Argentina in 1889, but educated in Europe after the age of fourteen. He is most famous for his short stories, though he was also a poet and an essayist, and in 1961 was awarded (jointly with Samuel Beckett) the International Publishers' Prize for *Ficciones*, a collection of short stories. Many of his works include mathematical and philosophical ideas. Specifically, Infinity and its mathematical (and often confusing) properties are used as metaphors in some of his stories, usually illustrating the absurdity of man's search for meaning in the infinite universe. We will explore how Borges uses some of the specific mathematical properties of Infinity metaphorically in a selection of his short stories.

The Book of Sand

In *The Book of Sand*, the narrator meets a stranger, who persuades him to exchange an antique Wyclif Bible for a fantastic book, called the Book of Sand. The book has some very strange properties that mirror those of any interval of the real number line $[a,b]$, where a is strictly less than b. For example, when asked to turn to the first or last page of the book, the narrator is unable.

> "I took the cover in my left hand and opened the book, my thumb and forefinger almost touching. It was impossible: several pages always lay between the cover and my hand. It was as though they grew from the very book." ([2], p. 481)

If we associate a with the front cover and b with the back cover, the pages correspond to the open interval (a,b). One of the properties of any such open interval is that it contains no smallest number; that is, there is no *first* number of the set of all number greater than a, but less than b. However close to a we choose a number x to be, there are still infinitely many numbers between a and x, namely all those in the interval (a,x).

Any interval (a,b) of the real line contains infinitely many numbers so the probability of selecting any particular number in the interval is 0. For the mathematical details, see [4], sections 1.6 and 1.7. For an intuitive explanation of this note that if we had ten pages, the probability of selecting any one would be 1 out of 10 or $.1$. If we had 100 pages, the probability would be $.01$. With 1000 pages it would be $.001$. As the number of pages goes up, the probability goes down. To make sense of the case of an infinite number of pages, mathematicians use the concept of a *limit*, and using this idea they can determine that with infinitely many pages, the probability of selecting any specific one is 0. For a rigorous proof, see [4], page 30.

This property of (a,b) is illustrated in the story when the narrator admires a drawing of an anchor on a certain page and the stranger tells him that he will never see that page again.

> "I took note of the page, and then closed the book. Immediately I opened it again. In vain I searched for the figure of the anchor, page after page." ([2], p 481)

Regardless of how small the portion of the book the narrator examines is, that portion is still analogous to an open interval (x,y), where x is less than y. This smaller interval is still an infinite set, and thus the probability of selecting the specific page from it is 0.

In the story, Borges illustrates another property of intervals (a,b), where a strictly less than b: Such a set is *uncountably infinite*. To explain the idea, we must first consider the set of counting numbers,

$$1,2,3,4,5,\ldots\ldots$$

This set is clearly infinite, but Cantor proved in 1865 that there is a hierarchy of infinities, indeed infinitely many of them, each larger than the last. The *cardinality* of a finite set is the number of elements in the set. *Cardinality* has a more technical definition when the number of elements in the set is infinite, but essentially it refers to the size of the set. The smallest infinity in Cantor's hierarchy is the *cardinality* of the set of counting numbers, which is called *aleph nought,* and any set with that number of elements is called *countably infinite*. The name refers to the fact that for any set of this size, there is a way to count the members of this set; not in finite time, but a scheme can be provided that will eventually count off each member of the set. The *cardinality* of both the real line and of any interval (a,b), is the same and is denoted by c. That is, there is a one-to-one correspondence between any interval (a,b) and the entire real line. It is shown that c is also the *cardinality* of the *power set*, the set of all subsets of integers. Therefore, c is in some sense much, much larger than *aleph nought*. Any set with *cardinality* higher than *aleph nought,* such as the real line or an interval (a,b), is called *uncountably infinite* since no counting scheme exists for such a set.

Since the pages of <u>The Book of Sand</u> correspond to an interval (a,b), a set that is not countable, there can be no systematic way to number the pages of the book. It is possible that Borges alludes to this property of (a,b) with the strange page numbering the narrator encounters as he examines the Book:

> "I was struck by an odd fact: the even-numbered page would carry the number 40,514, let us say while the odd-numbered page that followed it would be 999. I turned the page; the next page bore an eight-digit number." ([2], page 481)

Finally, after the narrator has become obsessed with the Book, he realizes that it has taken over his life. He decides to get rid of it, leaving it in a random place among the thousands of books in the National Library, where, we assume, it is lost forever.

The Library of Babel

The *Library of Babel* is not a story so much as a description of the world as a vast library and man as a librarian.

> "The universe (which others call the Library) is composed of an indefinite and perhaps infinite number of hexagonal galleries..." ([2], p. 112)

The story contains perhaps the largest number of mathematical ideas of any of Borges's stories. The very physical structure of the Library, with its arrangement of hexagonal galleries extending for as far as the eye can see, alludes to the idea of what is called a *tiling* of the *xy-plane*. A *tiling* of the plane is an arrangement of two-dimensional shapes placed in such a way, that their arrangement leaves out no part of the plane. A *regular tiling* is such an arrangement in which the shapes are all identical regular polyhedra. For example, it is fairly easy to imagine a *tiling* of the plane by squares; just imagine a piece of graph paper. It is also possible to *tile* the plane with equilateral triangles and with regular hexagons, but with no other regular polyhedra. A piece of honeycomb would be an example of a regular hexagonal *tiling*. *Tiling* is an interesting subject in mathematics, perhaps most famously explored by M.C.Escher who got many ideas from studying the mosaic patterns in the Alhambra.

As we can discuss two-dimensional *tilings* of the plane, we can also think about three-dimensional shapes that fit together in such a way that they fill up three-dimensional space. For example, it is fairly easy to see that cubes can stack together to fill up all of space. In mathematics this is the idea of *space-packings*. In the Library, the rooms seem to be hexagonal cylinders, which will also fill space, just as the chambers of honeycomb fill up the hive.

A more difficult idea lies in the statement,

> "The Library is a sphere whose exact center is any hexagon and whose circumference is unattainable." ([2], p. 113)

In any sphere of finite radius, there is a unique center point, namely the point that is equidistant from any point on the surface of the sphere. Since we are told that the Library is an infinite sphere, the distance from any point within to any point on the surface is infinite. In such a sphere, the set of points on the line connecting an interior point x to any point of the surface is an interval (x, ∞) with *cardinality* c. Since this is the case for all interior points, every point can be imagined in a transcendental way as the "center" of this imaginary sphere.

As in *the Book of Sand,* there is a reference to an infinite book, this time a circular book rumored to exist in a circular chamber. The book has "a continuous spine that goes completely around the walls....That cyclical book is God." ([2], p. 113)

It is very easy to relate the cyclical book found here, with the Book of Sand by using the mathematical analogy, and relating an interval of the real line to a circle. For example, the interval *[0,1]* can be related to the unit circle in the *xy-plane* parametrically using the parametrization,

$$[0,1) \to [\cos 2\pi t , \sin 2\pi t)$$

This gives a continuous, one-to-one mapping of the interval *[0,1)* onto the unit circle, where we have identified the front and back covers (the endpoints *0* and *1*.) Since the properties mentioned before about any interval *[a,b]* (hence about the Book of Sand) are preserved under a continuous, one-to-one mapping, those same properties will be true of the circle and hence of the circular book. The only difference is that there is no longer a cover as there are no longer endpoints.

In the story, we learn that there are twenty-five orthographical symbols (twenty-two letters, the coma, the period and the space) and that each book in the Library has four hundred and ten pages, each page containing forty lines, and each line containing eighty symbols. The Library contains every possible book. Indeed if we look at each book as simply a sequence of symbols four hundred and ten pages long, the Library contains every possible sequence of that length that can be constructed out of the twenty-five symbols. To determine this number, notice that the number of symbols in each sequence (book) is

$$410 \times 40 \times 80 = 1{,}312{,}000.$$

Since at any given position in the sequence, we can insert any of the twenty-five symbols, there are

$$25 \times 25 \times 25 \cdots\cdots \times 25 = 25^{1{,}312{,}000}$$

possible sequences, "a number which, though unimaginably vast, is not infinite." ([2], p. 115) Thus the Library contains

> "all that it is given to express, in all languages. Everything: the minutely detailed history of the future, the archangels' autobiographies, the faithful catalogue of the Library, thousands and thousands of false catalogues, the demonstrations of the fallacy of those catalogues, the demonstration of the fallacy of the true catalogue, the true story of your death, the translation of every book in all languages, the interpolations of every book in all books." ([2], p. 115)

Some believed it necessary to eliminate "useless works" but Borges notes

> "One: the Library is so enormous that any reduction of human origin is infinitesimal. The other: every copy is unique, irreplaceable, but (since the Library is total) there are always several hundred thousand imperfect facsimiles: works which differ only in a letter or a coma." ([2], p. 117)

While the number of different books in the Library is not infinite, Borges is touching upon the fact that removing or adding a finite number to an infinite number has no effect on the cardinality: the result is still infinite.

The famous paradox of Bertrand Russell is also alluded to in the story. A superstition that exists amongst the librarians:

> "on some shelf in some hexagon...there must exist a book which is the formula and perfect compendium *of all the rest*: some librarian has gone through it and he is analogous to god." ([2], p. 117)

This book which contains all books reminds us of Russell's question: Is the set of all sets a set? On one hand it is by definition. On the other, it cannot be since as it contains all sets, it must contain itself as one of its elements.

The story ends with the observation that since the Library is infinite yet there are only finitely many different possible books, the Library must be periodic in nature:

> "*The Library is unlimited and cyclical.* If an eternal traveler were to cross it in any direction, after centuries he would see that the same volumes were repeated in the same disorder...my solitude is gladdened by this elegant hope." ([2], p. 118)

The Garden of Forking Paths

In *The Garden of Forking Paths*, the narrator ponders a fragment of a letter,

> "I leave to the several futures (not to all) my garden of forking paths." ([2]. p. 125)

and eventually realizes that its author, Ts'ui Pen, was referring to forks in time, not in space.

> "In all fictional works, each time a man is confronted with several alternatives, he chooses one and eliminates the others; in the fiction of Ts'ui Pen, he chooses----simultaneously---- all of them. *He creates*, in this way, diverse futures, diverse times which themselves also proliferate and fork." ([2], p. 125)

The structure that is created by these forking paths is known in mathematics as a *tree* and is studied in a branch of mathematics known as *Graph Theory*. At each node of the *tree* there are a certain number of branches departing from that node, and at the end of each branch, there are further branches emanating outward. The process continues for finitely or infinitely many steps. In the story, each node is a moment when a choice is made. Each choice is followed until another decision is required, again involving a number of choices. Each of those choices is followed until another decision is required, and so on. For example, upon waking I could choose between going running, going bicycling or doing neither. I then might choose between a shower or a bath, and finally whether to have tea or coffee. In the Garden, each choice is made and all possibilities exist at once. With these three decisions, there are already twelve possible realities. For example, in one I went running, showered and had tea. In another I got no exercise, bathed and had tea.

A *path* in the *tree* is a collection of consecutive branches (or edges) of the *tree*. If we impose a direction on the graph that corresponds with the direction of positive time, and only look at *paths* that flow in that direction, each of these *paths* in the Garden corresponds to a different possible life. In the very small *tree* in my example, each of the two possibilities I listed would be an example of such *paths*.

To make things simpler, suppose that at each node, there were only two choices. So with three levels, we would have $2{\times}2{\times}2{=}8$ *paths*. With four steps we would have $2{\times}2{\times}2{\times}2{=}16$ *paths*. With n steps, we would have

$$2\times 2\times 2\cdots\cdots\times 2 = 2^n$$

paths. So the number of possible *paths* (lives) grows exponentially with the number of steps. In Borge's story, it is assumed that there are infinitely many such steps (decisions.)

> "He (Ts'ui Pen) believed in an infinite series of times, in a growing, dizzying net of divergent, convergent and parallel times." ([2], p. 127)

By assuming this, Borges is again touching upon the hierarchy of infinities alluded to earlier. With any one of our *paths*, the set of steps is necessarily *countably infinite*, since the decisions occur in sequence in time, and thus can be ordered. But if at each step there are, for example, even only two choices, then the set of all possible *paths* is *uncountably infinite* because it can be shown that the *cardinality* of this set is c, the *cardinality* of both the real line and of any interval (a,b) , where a is strictly less than b.

The Circular Ruins

The Circular Ruins is a story modeled on the mathematical idea of *recursion*. In the story, a man arrives by boat in a strange land, wounded and exhausted. He lies beside the ruins of a temple and falls asleep, not simply to rest, but for a higher reason.

> "The purpose which guided him was not impossible, though it was supernatural. He wanted to dream a man: he wanted to dream him with minute integrity and insert him into reality."
> ([2], p. 97)

The man continues dreaming night after night, struggling with his task until one night he "dreamt of a beating heart." He spends several nights perfecting the heart and then moves on to the arteries, other organs, skeleton until finally he dreams each hair.

> "he dreamt a complete man, a youth, ...night after night the man dreamt him as asleep."
> ([2], p. 98)

Then one night with the help of a fire god,

> "In the dreamer's dream, the dreamed one awoke." ([2], p. 99)

The man continues dreaming, teaching through his dreams his newly conscious creation how to do various things, preparing the youth for reality. Just before the youth is ready to be born

> "(so that he [the dreamed one] would never know he was a phantom, so that he would be thought of as a man like others) he [the dreamer] instilled in him a complete oblivion of his years of apprenticeship."
> ([2], p. 99)

Time passes after the man has fulfilled "his life's purpose." When he feels his own death coming, he walks into a fire, but the flames

> "did not bite into his flesh, they caressed and engulfed him without heat or combustion. With relief, with humiliation, with terror, he understood that he too was a mere appearance, dreamt by another." ([2], p. 100)

Thus Borges tells us of a man who dreams another man into existence eventually realizing that he is but the dream of another. This implies an infinite chain stretching in both directions:

....a man who dreams a man, who dreams a man, who dreams a man, who....,

and is a beautiful, literary example of *recursion*.

Other Examples of Infinite Recursion

The theme of infinite *recursion* occurs in other writings of Borges. In the short story *The God Script,* a man imprisoned in a cell, dreams a grain of sand one night. The next night he dreams two grains. His dreaming continued until eventually, the cell was filled with sand. When he finds himself suffocating in his cell, he desperately tries to wake himself, but a voice says:

> "You have wakened not out of sleep, but into a prior dream, and that dream lies within another, and so on, to infinity, which is the number of grains of sand." ([2], p. 252)

Perhaps the most famous example of infinite *recursion* in Borges's writings (and the most often quoted) is in the last lines of his poem *Chess II*:

> "God moves the player, he in turn the piece.
> But what god beyond God begins the round
> Of dust and time and sleep and agonies?"

Conclusion

As these four stories demonstrate, mathematical ideas are essential to the poetic thoughts and imaginings of Borges. In particular, he elegantly uses many of the properties of infinity in his short stories. Here they confound and dazzle his characters, as they have so often confounded and dazzled us.

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