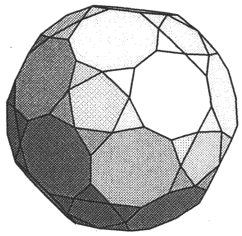
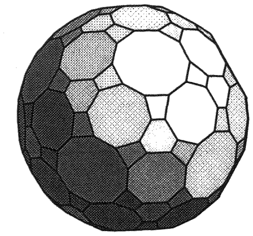
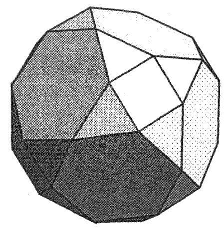
BRIDGES
Mathematical Connections
in Art, Music, and Science



Symmetrohedra: Polyhedra from Symmetric Placement of Regular Polygons



Craig S. Kaplan
University of Washington
Box 352350, Seattle, WA 98195 USA
csk@cs.washington.edu



George W. Hart
http://www.georgehart.com
george@georgehart.com

Abstract

In the quest for new visually interesting polyhedra with regular faces, we define and present an infinite class of solids, constructed by placing regular polygons at the rotational axes of a polyhedral symmetry group. This new technique can be used to generate many existing polyhedra, including most of the Archimedean solids. It also yields novel families of attractive symmetric polyhedra.

1 Introduction

Most interesting polyhedra arise via a process that attempts to generate or preserve some measure of order. We might ask that the polyhedron be as symmetric as possible, that the symmetries act transitively on its vertices, edges or faces, or that all its faces be regular polygons. Indeed, these questions have all been asked and the resulting families of polyhedra enumerated to satisfaction. The last case, that of convex polyhedra all of whose faces are regular polygons, are the well-known Johnson solids [3].

Eager to discover new polyhedra with regular faces, but aware that the list of Johnson solids has been proven complete, we are faced with the prospect of letting some constraints slip a bit in order to innovate. We achieve this innovation by dropping the constraint that all faces be regular, asking only that *many* faces be regular, and that the remaining faces occur in a small number of different shapes.

In this paper we define an infinite set of convex polyhedra that we call *symmetrohedra*. They are parameterized by several discrete and continuous values. All symmetrohedra have the "many regular faces" condition prescribed above, and for many choices of parameters the number of remaining face shapes is small. Furthermore, every symmetrohedron has full tetrahedral, cuboctahedral, or icosidodecahedral symmetry, giving it a pleasing global order beyond the regularity of its faces.

We proceed by giving an example of how a symmetrohedron is constructed, followed by a rigorous definition of the class and some implementation notes. We then take a tour of the world of symmetrohedra, organized into groups with related properties. While a few isolated examples go back to the Renaissance (*e.g.*, the middle example above appears in Barbaro [1]), most of the 71 polyhedra illustrated here are new.

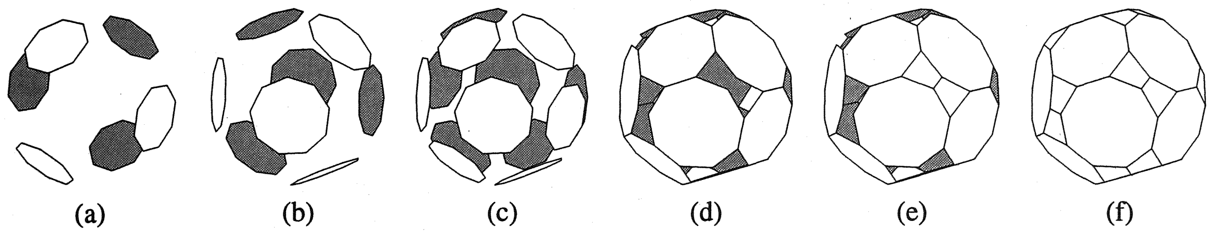


Figure 1 Steps showing how the symmetrohedron $\mathcal{O}(2,3,*,\mathrm{e})$ is constructed.

2 An Example

Figure 1 shows an example of how a cuboctahedral symmetrohedron is constructed. The cuboctahedral symmetry group has fourfold axes, threefold axes, and twofold axes. A cube, for example, has these axes passing through its face centres, vertices, and edge midpoints, respectively. Here, we will make use of the fourfold and threefold axes to place faces of a new polyhedron. In step (a), an octagon is centred on both ends of every fourfold axis and aligned perpendicular to it. In step (b), an enneagon (of equal edge length) is placed at both ends of every threefold axis. Step (c) shows the two together. We call these faces the *axis-gons* of the resulting solid. Note that the number of sides of an axis-gon must be a multiple of the degree of the axis, to ensure that the face does not break any rotational symmetries. Furthermore, to preserve mirror symmetries, the face can rotate about its axis into one of only two positions; the planes of reflection meeting at the axis can pass through the vertices of the face or the midpoints of its edges. Here, the faces are rotated so that the planes pass through their edges, which also causes edges of adjacent axis-gons to be parallel.

If we slide the octagons and enneagons along their axes, there will come a point where the two nearest parallel edges of neighbouring faces will mate. Because of symmetry, this mating will happen simultaneously for all axis-gons, resulting in the object shown in step (d). This object is very nearly a solid, but it has twelve bowtie-shaped holes, one centred on every twofold cuboctahedral axis. To make it into a solid, we need to create new faces that fill these holes. The simplest way to do so is shown in step (e); a hole is filled with two congruent trapezoids, joined by a ridge between two enneagonal vertices. Because all the holes are congruent, they can all be filled with identical pairs of trapezoids, resulting in the completed symmetrohedron of step (f), an "octahedral bowtie symmetrohedron". This solid has many regular faces (including enneagons, which appear seldom in the literature on polyhedra), and all remaining faces are of a single shape.

3 Definition and Notation

The process described above generalizes naturally in several directions. We can start with the rotational axes of any of the three non-prismatic polyhedral symmetry groups: $\mathcal{T}$, the tetrahedral group, $\mathcal{O}$, the cuboctahedral group, and $\mathcal{I}$, the icosidodecahedral group. We can choose which sets of axes to use for placement of axis-gons. For each axis set, we can pick any regular polygon whose number of sides is a multiple of the degree of the axes in that set. And we can choose whether the axis-gons should be rotated so that they mate edge-to-edge as above or vertex-to-vertex.

Each symmetrohedron is described by a concise symbolic expression $G(l, m, n, \alpha)$. G represents the choice of group; it is one of $\mathcal{T}$, $\mathcal{O}$, and $\mathcal{I}$. The values l, m and n are the *multipliers*; a multiplier of m will cause a regular km-gon to be placed at every k-fold axis of G. In the notation, the axis degrees are assumed to be sorted in descending order, *i.e.*, 5-, 3- and 2-fold for $\mathcal{I}$, 4-, 3-, and 2-fold for $\mathcal{O}$, and 3-, 3-,

$\mathcal{T}(1,*,*,\mathsf{e})$	tetrahedron	$\mathcal{O}(1,*,*,\mathsf{e})$	cube
$\mathcal{O}(*,1,*,\mathsf{e})$	octahedron	$\mathcal{I}(1,*,*,\mathsf{e})$	dodecahedron
$\mathcal{I}(*,1,*,\mathsf{e})$	icosahedron	$\mathcal{O}(1,1,*,\mathsf{e})$	cuboctahedron
$\mathcal{I}(1,1,*,\mathsf{e})$	icosidodecahedron	$\mathcal{T}(2,1,*,\mathsf{e})$	truncated tetrahedron
$\mathcal{O}(2,1,*,\mathsf{e})$	truncated cube	$\mathcal{O}(1,2,*,\mathsf{e})$	truncated octahedron
$\mathcal{I}(2,1,*,\mathsf{e})$	truncated dodecahedron	$\mathcal{I}(1,2,*,\mathsf{e})$	truncated icosahedron
$\mathcal{O}(1,1,*,1)$	rhombicuboctahedron	$\mathcal{I}(1,1,*,1)$	rhombicosidodecahedron
$\mathcal{O}(2,2,*,\mathsf{e})$	truncated cuboctahedron	$\mathcal{I}(2,2,*,\mathsf{e})$	truncated icosidodecahedron

Figure 2 Examples of how all but two of the Platonic and Archimedean solids may be realized as symmetrohedra.

and 2-fold for $\mathcal{T}$. We also allow two special values for the multipliers: $*$, indicating that no polygons should be placed on the given axes, and 0, indicating that the final solid must have a vertex (a zero-sided polygon) on the axes. We require that one or two of l, m, and n be positive integers.

The final parameter, α, controls the relative sizes of the non-degenerate axis-gons. When two multipliers are positive integers, α is the ratio of the side length of the first axis-gon to the side length of the second axis-gon, in the order given in the symbol. Obviously, this ratio can only vary when the axis-gons mate vertex-to-vertex. We use the special symbol e in this position when the axis-gons should be rotated to mate edge-to-edge, in which case they will always have the same side length.

Note that if one set of axis-gons is scaled up enough relative to the other, eventually they will mate with each other. Scaling the other way causes the other set of axis-gons to mate. Rather than write the actual values of α where this happens, we use the shorthand $[k]$ to indicate that the axis-gons in the k-th position of the symbol should be scaled until they mate with each other.

We formally define a symmetrohedron to be a polyhedron that corresponds to a legal instance of this notation. The example given in the previous section can be written as $\mathcal{O}(2,3,*,\mathsf{e})$.

4 Implementation

We have created a computer program that accepts as input the symbol of a symmetrohedron and produces a three-dimensional representation of the solid as output. The implementation derives, in closed form, the placement of all the axis-gons, obviating the need for the sliding procedure in the example. The program is written in roughly two thousand lines of C++, and runs in about a second on a low-powered PC. It outputs a description of the solid compatible with a 3D OpenGL-based visualization program. We modified the program to produce the renderings of the solids seen in this paper.

Another implementation detail is the mechanism by which any holes are filled in. For many low-valued choices of multipliers, a hand-coded set of faces may suffice, but an automatic solution is desirable in general. Our automatic solution takes the 3D convex hull[4] of all the axis-gon vertices. Except for the kite cases defined below, the initial vertices are inscribable in a sphere, so the convex hull will have the same set of vertices, preserve all existing faces, and create new ones only where there were holes. This behaviour corresponds to our choice of hand-coded hole fillers in every case.

5 A Survey of Symmetrohedra

We take a brief tour of the many interesting polyhedra that can be described using this system.

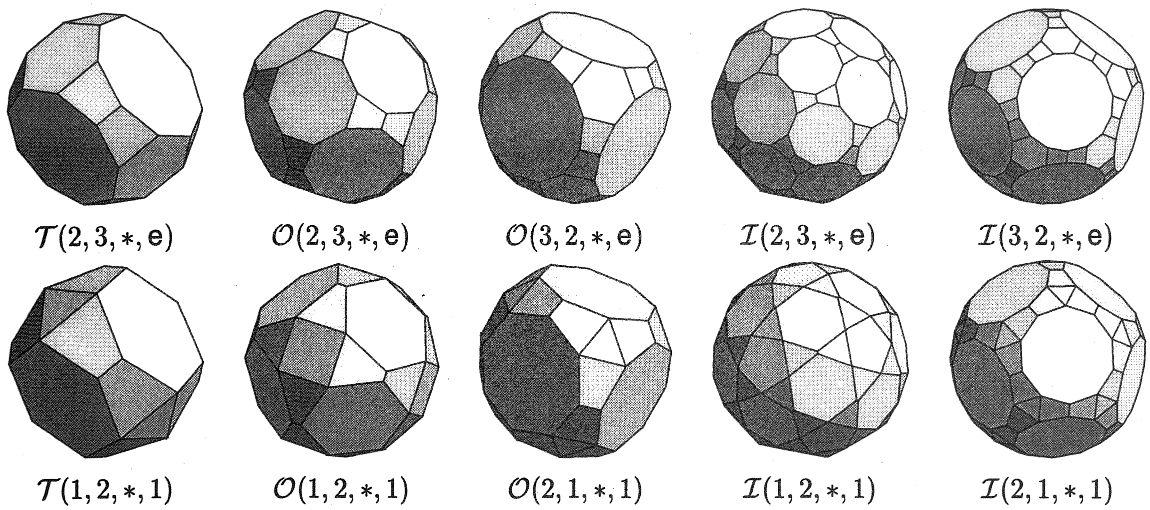


Figure 3 The complete set of bowtie symmetrohedra (cravatahedra?).

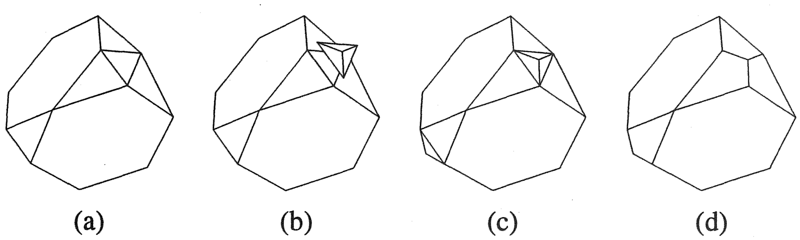


Figure 4 The construction of the kite-filled symmetrohedron $\mathcal{T}(2,0,*,[1])$.

5.1 Well Known Polyhedra.

Of the eighteen Platonic and Archimedean solids, only the two snubs do not have complete $\mathcal{T}$, $\mathcal{O}$, or $\mathcal{I}$ symmetry. The other sixteen are all reproducible as symmetrohedra, as shown in Figure 2. In many cases, several different symbols can be given for an Archimedean solid; for simplicity, we show just one.

5.2 Bowtie Symmetrohedra.

In Section 2, the filled holes are said to be bowtie-shaped. Holes of this shape are defined by six edges, two belonging to one kind of axis-gon and four belonging to another. These must appear on the twofold axes. In the edge-mated case, we obtain such holes when $(l,m) = (2,3)$ or $(3,2)$. In the vertex-mated case, we must have $(l,m) = (1,2)$ or $(2,1)$. Because the tetrahedral group is symmetric in l and m, we arrive at a grand total of ten *bowtie symmetrohedra*, as shown in Figure 3.

5.3 Kite-Filled Symmetrohedra.

In Figure 4(a), we see the symmetrohedron $\mathcal{T}(2,*,*,[1])$. the holes in this case are shaped like non-planar three-pointed stars, each one an equilateral triangle surrounded by three skinny isoceles triangles. If we affix a pyramid of just the right height to the central triangle, the sides of the pyramid will be coplanar with the isoceles neighbours, creating three kite shapes. The resulting solid, $\mathcal{T}(2,0,*,[1])$, is shown in (d). In general, when a hole is surrounded symmetrically by congruent axis-gons, and each one has a vertex pointing into the hole, it is possible to create a *kite-filled symmetrohedron*. As with the bowtie symmetrohedra, there are ten of these solids, shown in Figure 5.

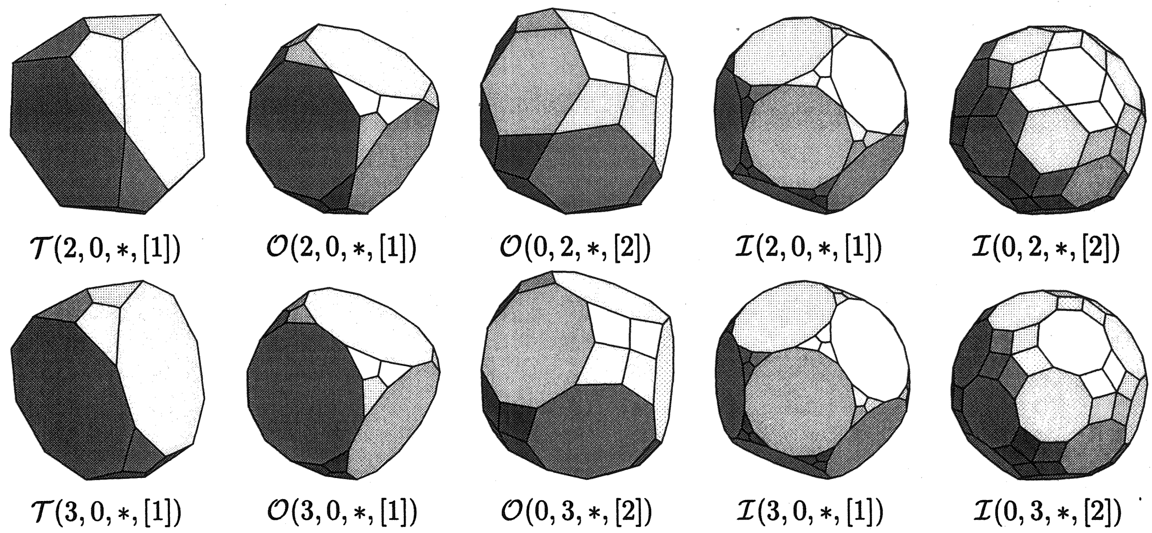


Figure 5 The complete set of kite-filled symmetrohedra.

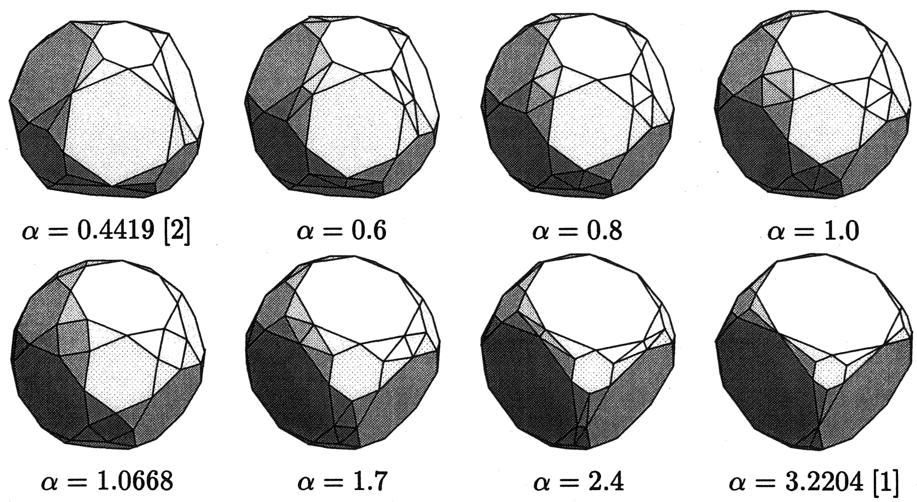


Figure 6 A demonstration of the effect of varying edge-length ratio, α, in $\mathcal{O}(2, 2, *, \alpha)$.

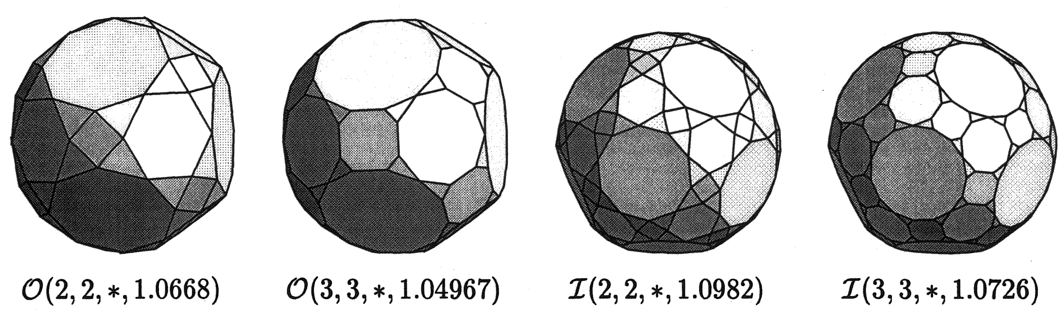


Figure 7 Symmetrohedra for which varying α can coalesce hole-filling polygons into larger planar regions. The first appears in Jamnitzer [2].

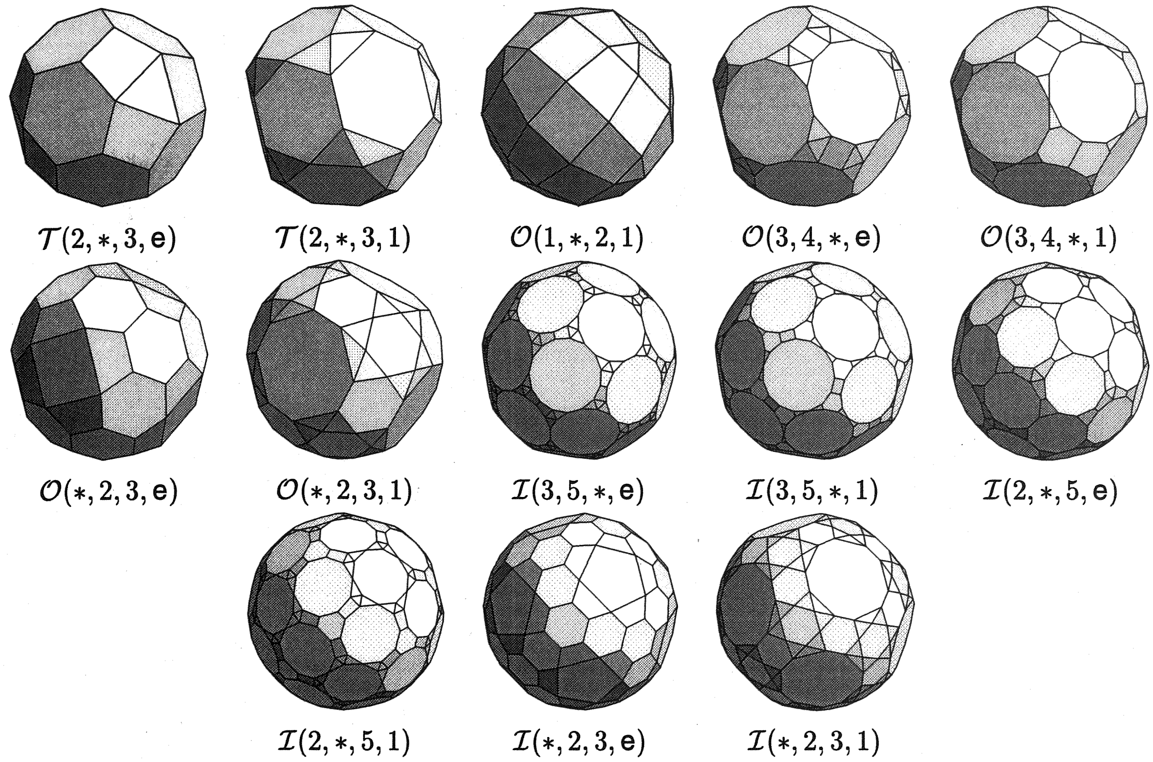


Figure 8 A set of LCM symmetrohedra.

5.4 Changing α and Planar Rhomb Polyhedra.

As described in Section 3, there is an additional continuous parameter α in the vertex-mated case that controls the ratio between the side lengths of the two axis-gons. As α varies in a symmetrohedron, a continuous family of polyhedra evolves, with occasional transitions in topology.

Figure 6 gives a demonstration of changing α gradually from $0.4419 = [2]$ to $3.2204 = [1]$. An interesting crossover point happens at $\alpha = 1.0668$. At this point, the relative sizes are balanced so that the protuding hexagonal and octagonal vertices lie in the same plane. The hole's central isoceles triangles flatten out into rhombs. Figure 7 gives some other examples of this α-dependent flattening behaviour.

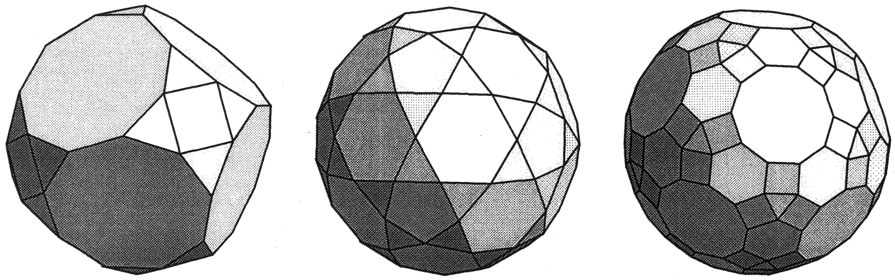
5.5 LCM Symmetrohedra.

A particularly appealing set of polyhedra occurs when the multipliers are chosen so that two families of axis-gons have the same number of sides. The number of sides will be a multiple of the Least Common Multiple of the two axis degrees, from which we get the name *LCM symmetrohedra.* A collection of low-order LCM symmetrohedra appears in Figure 8.

5.6 Near Misses.

The classification of Johnson solids led to the unusual cases of *near misses*, polyhedra whose faces are so close to regular that they can often be physically constructed without noticing the discrepancy.

To our knowledge, there is no systematic analysis of near misses; they are discovered by trial and error. The study of symmetrohedra does not eliminate the guesswork from the discovery of new near misses. Nevertheless, because our stated goal was to create polyhedra with many regular faces, the near misses do tend to present themselves readily.



$\mathcal{O}(*, 3, *, [2])$ $\mathcal{I}(1, 2, *, [2])$ $\mathcal{I}(2, *, 3, \mathtt{e})$

Figure 9 Three interesting near misses. The second is found in Barbaro [1].

Three near misses are given in Figure 9. Presumably these were all considered previously by researchers seeking Johnson polyhedra. The third is constructed from regular decagons, hexagons, and triangles, and trapezoids with internal angles of approximately 88.3° and 91.7°.

5.7 A Sampling of Icosidodecahedral Symmetrohedra.

We end our tour at Figure 10 with twenty other interesting symmetrohedra that have not yet made an appearance. All are based on the icosidodecahedral symmetry group.

6 Conclusion

Symmetrohedra provide a range of attractive symmetric forms that feature regular polygons as faces. We hope some may provide inspiration to sculptors, architects, puzzle and toy designers, and other hobbyists and professionals.

Many natural generalizations not considered here are also possible, such as eliminating mirror symmetry and introducing face rotation parameters, adding star-shaped and other non-convex faces, allowing the axis-gons to not mate with neighbours, introducing additional regular polygons in arbitrary off-axis positions and replicating them via the symmetry group, or experimenting with prism symmetries and self-intersecting variations.

Acknowledgments

We would like to thank Douglas Zongker for providing us with software for visualizing and rendering polyhedra. His software was used in part to prepare all the diagrams in this paper. Thanks also to the members of the polyhedron mailing list for feedback and comments on early examples.

References

[1] Daniele Barbaro. *La Pratica Della Perspettiva*. 1569. Arnaldo Forni reprint, 1980.

[2] Wentzel Jamnitzer. *Perspectiva Corporum Regularium*. Nuremburg, 1568. (Siruela reprint, Madrid, 1993). `http://www.georgehart.com/virtual-polyhedra/jamnitzer.html`.

[3] Norman W. Johnson. Convex solids with regular faces. *Canadian Journal of Mathematics*, 18:169–200, 1966.

[4] Joseph O'Rourke. *Computational Geometry in C*. Cambridge University Press, 2 edition, 1998.

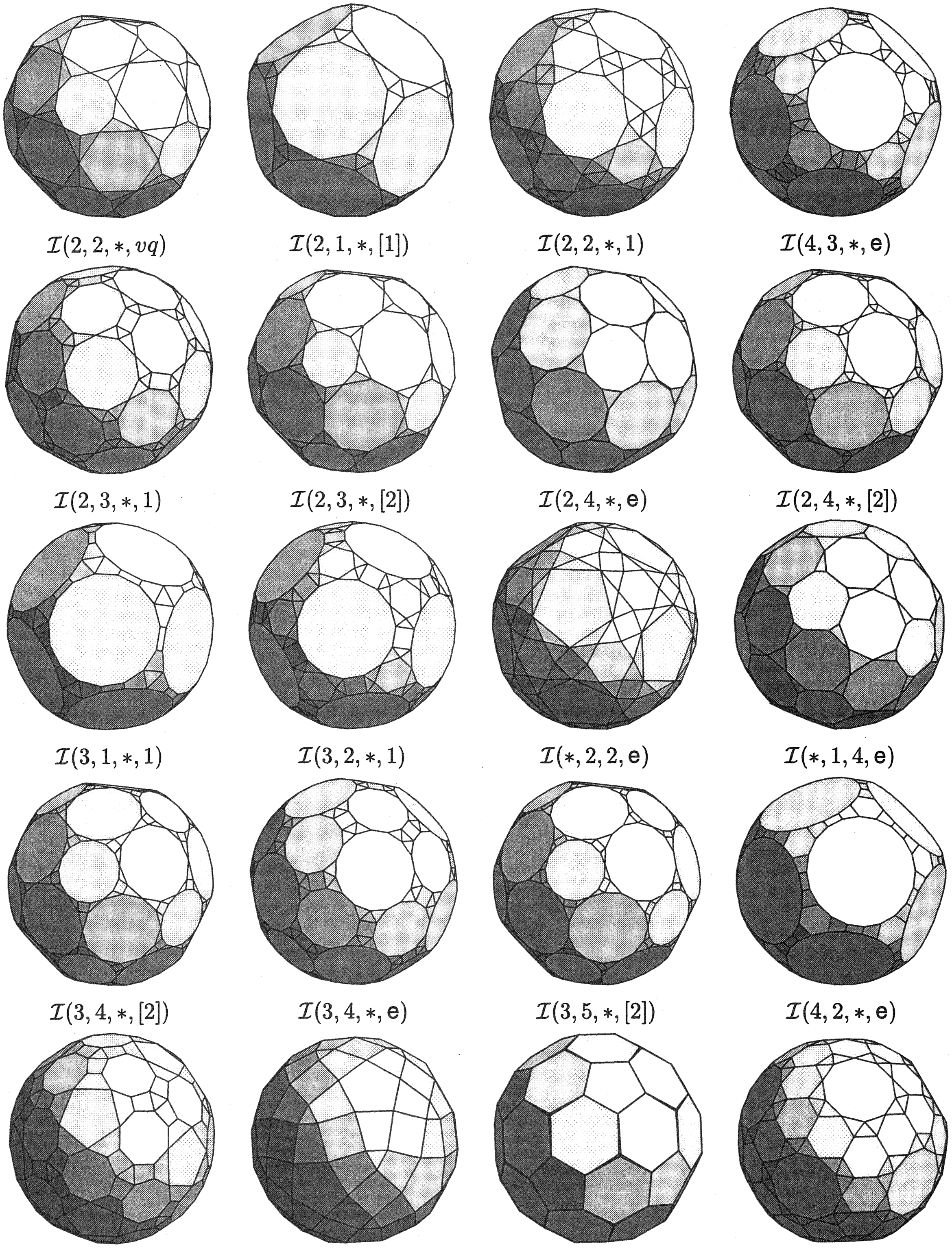


Figure 10 Appealing symmetrohedra with $\mathcal{I}$ symmetry. $\mathcal{I}(2,4,*,\mathsf{e})$ is a twice-truncated icosahedron, found in Barbaro [1] along with $\mathcal{O}(2,4,*,e)$. $\mathcal{I}(1,*,2,1)$ appears in Jamnitzer [2]. $\mathcal{I}(1,*,3,\mathsf{e})$ is related to a rhombic triacontrahedron with its fivefold vertices truncated.