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On the Construction of Colored Plane Crystallographic Patterns

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Abstract

In this paper, we present an approach to the construction of perfect and non-perfect colorings resulting from plane crystallographic groups. In particular, we consider colored patterns that arise with symmetry group normal in the symmetry group of the uncolored pattern.

Keywords: colored symmetrical patterns, perfect colorings, non-perfect colorings, plane crystallographic groups

Introduction

In the theory of color symmetry, one problem of interest is the study and analysis of colored symmetrical patterns. There are two types of colorings of a symmetrical pattern. If G is the symmetry group of the pattern with the colors disregarded, the pattern is said to be perfectly colored if every element of G affects a permutation of the colors of the pattern. In those instances when not all elements of G permute the colors of the pattern, we obtain a non-perfectly colored pattern.

To illustrate these two types of colorings, let us consider the colored patterns appearing in Figure 1 which are assumed to repeat over the entire plane. For both, the symmetry group G of the patterns with the colors disregarded is the plane crystallographic group *p6m* generated by the 60°counterclockwise rotation r about the indicated point p, the reflection s in the horizontal line through p and the translations x,y. (see Figure 3). The pattern in Figure 1(a) is perfectly colored since every element of G affects a permutation of the colors. On the other hand, the pattern appearing in Figure 1(b) is not perfectly colored since there are elements of G that do not permute the colors. For instance, applying the reflection s will send the color grey to the colors black and white. In fact, for this colored pattern, the elements of G permuting the colors belong to the set generated by r, x and y which form a subgroup of G of plane crystallographic type *p6*.

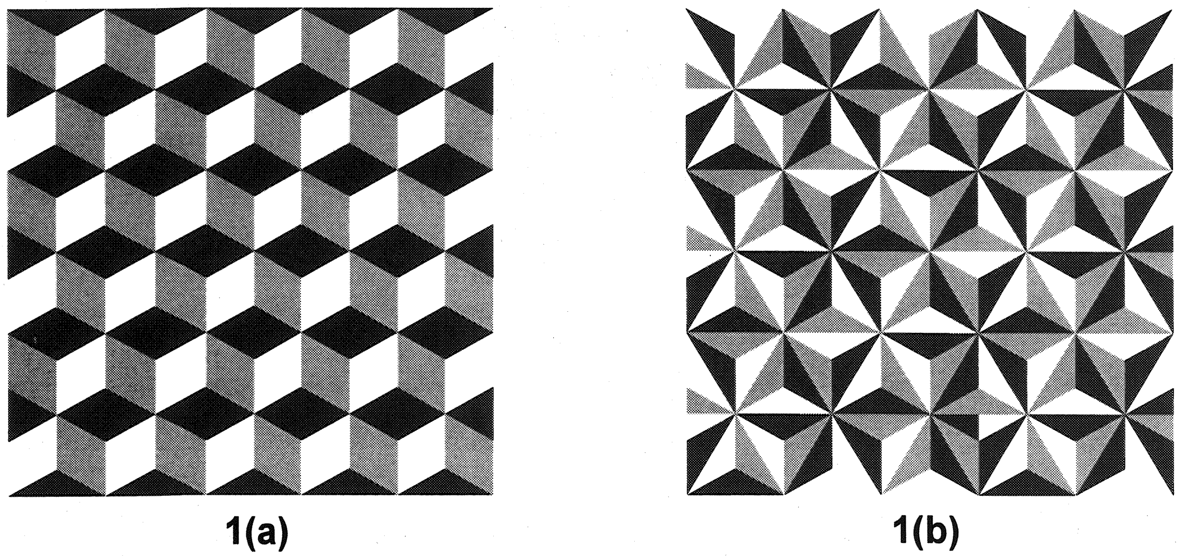


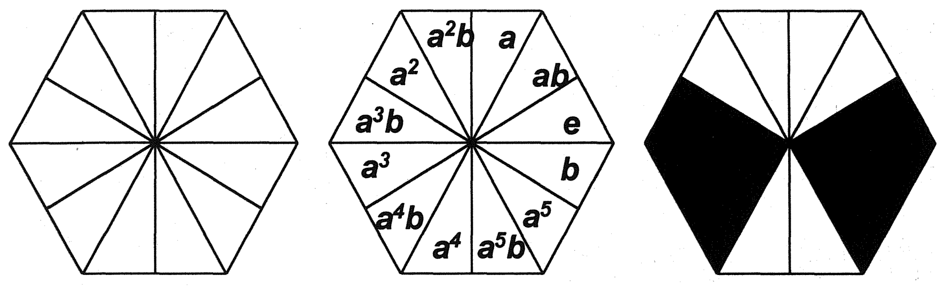
Figure 1: **1(a) perfectly-colored pattern; 1(b) non-perfectly-colored pattern**

The purpose of this note is to illustrate the construction of colored plane crystallographic patterns, which include the perfectly and non-perfectly colored ones. The approach we consider here is based on a framework for analyzing colored symmetrical patterns which was discussed in detail in [1] and [2].

Preliminaries

Let us now describe the setting in which we will work with colorings.Let G be the symmetry group of an uncolored pattern where G is a plane crystallographic group or a subgroup of a plane crystallographic group. By a plane crystallographic group we refer to the group of isometries of the Euclidean plane whose translations form a subgroup which is a free abelian group of rank two. A subgroup of a plane crystallographic group is either a plane crystallographic group, a frieze group or a finite group which is cyclic or dihedral. A frieze group is a group of isometries of the Euclidean plane whose translations from a subgroup which is an infinite cyclic group. Now consider a subset S of a fundamental domain for G. The set $\{g(S) : g \in G\}$ is called the G-orbit of S. Our assumption is that the given pattern can be obtained as the G-orbit of some subset S of a fundamental domain for G. This G-orbit of S and G are in one-to-one correspondence under the rule $g(S) \leftrightarrow g$ for each $g \in G$, so that each element of the G-orbit may be labeled by each element of G. By assigning a color to each element of G, we assign a color to each set $g(S)$. This assignment of colors is called a **coloring** of the pattern. This results in a partition P of G where a set in P consists of elements assigned the same color so that a coloring is simply a partition of G.

To illustrate the above concept of a coloring let us consider the uncolored pattern V appearing in Figure 2(a) which has symmetry group $G = D_6 = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$ where a is a 60°-counterclockwise rotation about the center of the hexagon and b is a reflection in the horizontal line through the center of the hexagon. If S is the triangular region labeled "e" in Figure 2(b) then for each $g \in G$, the triangular region $g(S)$ is labeled "g". Given the following partition of G, $\{e, b, a^5, a^3b, a^3, a^4b\}$ and $\{a, a^2, a^4, ab, a^2b, a^5b\}$ to which we assign the colors black and white respectively, we obtain the non-perfect coloring in Figure 2(c).



(a) (b) (c)

Figure 2: **2(a) uncolored pattern** V **with symmetry group** D_6; **2(b) the labelled triangular regions; 2(c) a non-perfect coloring of** V

In the analysis of a coloring, three groups play a significant role. These groups are:

G = the symmetry group of the uncolored pattern
H = the subgroup of elements of G which permute the colors
K = the subgroup of elements of G which fix the colors

We will refer to H as the subgroup of color transformations and K as the symmetry group of the colored pattern. The groups G, H and K are such that $K \leq H \leq G$. If a group G permutes the colors of the pattern, that is $H = G$, then the coloring is perfect. Given a color, its stabilizer in G will lie between H and K. Since H acts on the set C of colors of the pattern, this action induces a homomorphism $f : H \to A(C)$, where $A(C)$ is the group of permutations of the set C of colors of the pattern. For $h \in H$, $f(h)$ is the permutation of the colors that h induces. An element h is in the kernel of f if and only if $f(h)$ is the identity permutation, that is, h fixes all the colors. Thus the kernel of f is K and the resulting group of color permutations $f(h)$ is isomorphic to H/K. Consequently, K is a normal subgroup of H.

If we treat a coloring as a partition $P = \{P_i : i \in I\}$ of a group G, then $H = \{g \in G : (\forall i \in I)(\exists j \in J)(gP_i = P_j)\}$ and $K = \{g \in G : (\forall i \in I)(gP_i = P_i)\}$.

Enumerating Colorings associated with Plane Crystallographic Patterns

In [1] and [2], a framework was presented for analyzing colored symmetical patterns. Moreover, the framework allowed for the listing of colorings for an uncolored pattern with symmetry group G and subgroups H, K of G such that $K \leq H \leq N_G(K)$, where the elements of H permute the colors and the elements of K fix the colors. In this note, we will adapt this framework to give rise to our construction of colored plane crystallographic patterns.Before we proceed to present our main results, we mention the highlights discussed previously in [1] and [2] which are important points for consideration. These concepts form the basis for the method used in coloring symmetrical patterns.

The assumptions we are to consider in determining colorings will be as follows. Let G be a group and H a subgroup of G. Let P be a partition of G. Since a partition of G corresponds to a coloring, we refer to P as the set of colors.

Definition 1. *Let G be a group, $H \leq G$, Y a complete set of right coset representatives of H in G, $\bigcup_{i\in I} Y_i$ a decomposition of Y and for each $i \in I$, $J_i \leq H$. Then the coloring or decomposition $G = \bigcup_{i\in I} \bigcup_{h\in H} hJ_iY_i$ or the* *partition of* G, $P = \{ hJ_iY_i : i \in I, h \in H\}$ *is called a* (Y_i,J_i)*-*H *coloring.*

Lemma 2. *A* (Y_i,J_i)*-*H *coloring of* G *defines an* H*-invariant partition of* G.

Remark 3. Also, if $K \leq G$ such that $H \leq N_G(K)$ and $K \leq J_i$ for each i, then the elements of K fix each of the sets hJ_iY_i because if $k \in K$ then $khJ_iY_i = hk^*J_iY_i = hJ_iY_i$.

Lemma 4. *If* $P = \{P_i : i \in I\}$ *is a* G*–invariant partition of the group* G, *then* P *is the partition of* G *consisting of left cosets of some subgroup* S *of* G. *This subgroup is the set in the partition containing* e. *Moreover, the subgroup of elements of* G *fixing* $P = \{P_i : i \in I\}$ *is* $core_G S$.

Lemma 5. *Let* G *be a group,* X *a non-empty subset of* G *and* K *a subgroup of* G. *Then* $kX = X$ *for all* k *in* K *if and only if* X *is a union of right cosets of* K *in* G.

Theorem 6. *Let* G *be a group and* H *a subgroup of* G. *If* P *is an* H*-invariant partition of* G *then* P *corresponds to a decomposition of* G *in the form* $G = \bigcup_{i\in I}\bigcup_{h\in H} hJ_iY_i$ *where* $\bigcup_{i\in I} Y_i = Y$ *is a complete set of right coset representatives of* H *in* G *and* $J_i \leq H$ *for every* $i \in I$. *If in addition* $K \leq H$ *and* K *fixes the elements of* P, *then* $K \leq J_i$ *for every* $i \in I$.

The above theorem characterizes all partitions of a group G which are invariant under multiplication on the left by elements of a subgroup H of G and whose elements are left fixed by multiplication on the left by elements of a subgroup K of H. It should be mentioned that distinct complete sets of coset representatives of H in G may give rise to the same partition. This situation was discussed in [1].

For our main results in this paper, we will determine the H-invariant partitions that arise from a given plane crystallographic group G which is the symmetry group of an uncolored pattern where the elements of K fix the colors such that $K \leq H \leq N_G(K)$ and K is a normal subgroup of G.

The assumption regarding the normality condition imposed on the subgroup K of G allows us to form the quotient group of G by K, denoted by G/K from which helpful information can be obtained in characterizing the colorings arising from G. It turns out that the construction of the perfect and non-perfect colorings associated with the given groups G, H and K is influenced by the group structure of G/K, for instance whether it is cyclic or dihedral.

A certain number of the colorings which are non-perfect may be considered equivalent. To determine if two colorings corresponding to two different partitions of G are equivalent we use the following definition.

Definition 7. *Consider the partitions* P,Q *of a group* G *which correspond to colorings* C *and* C' *respectively. The colorings* C *and* C' *are equivalent if and only if there exists a* $g \in G$ *such that* $Q = gP$.

We now give our main results below. We consider the particular cases when $[G : H] = 2, 3,$ or 4 and G/K is cyclic or dihedral of at most twelve elements.

Theorem 8. *Let* G *be a plane crystallographic group and* $H,K \leq G$ *where* K *is normal in* G. *Let* $G = \bigcup_{i\in I}\bigcup_{h\in H} hJ_iY_i$ *be a* $(Y_i,J_i) - H$ *coloring satisfying Theorem* 6. *There are four perfect and four non-perfect such colorings that arise if* G/K *is the cyclic group of order* 6, *denoted by* Z_6 *and* $[G : H] = 2$. *Moreover, the equivalent non-perfect colorings come in pairs.*

Proof. *Let* $G/K = \{K, aK, a^2K, a^3K, a^4K, a^5K\}$ *be the cyclic group* Z_6 *of order* 6. *The proper subgroups of* G *may be described as* $H_1 = K \cup a^2K \cup a^4K$ *and* $H_2 = K \cup a^3K$. *Since* $[G : H] = 2$, *we let* $H = H_1$. *Under* *the action of H on the set of right cosets of K in G, K\G, by left multiplication we get two orbits of right cosets,* $\{K, Ka^2, Ka^4\}$ *and* $\{Ka, Ka^3, Ka^5\}$. *Note that K is normal in G, so that every left coset is a right coset of G. Using Theorem* 6 *we obtain Table* 1 *where the colors* 1,2,...,6 *are assigned to the right cosets of K in G. There are* 8 $(J_i, Y_i) - H_1$ *colorings obtained.*

	H			Ha				
	K	Ka^2	Ka^4	Ka	Ka^3	Ka^5	J_i and Y_i used	
C_1	1	1	1	1	1	1	$J_1 = H$; $Y_1 = \{e, a\}$	G
C_2	1	1	1	2	2	2	$J_1 = J_2 = H$ $Y_1 = \{e\}; Y_2 = \{a\}$	H_1
C_3	1	1	1	2	3	4	$J_1 = H; J_2 = K$; $Y_1 = \{e\}; Y_2 = \{a\}$	N-PC
C_4	1	2	3	4	4	4	$J_1 = K; J_2 = H$; $Y_1 = \{e\}; Y_2 = \{a\}$	N-PC
C_5	1	2	3	4	5	6	$J_1 = J_2 = K$; $Y_1 = \{e\}; Y_2 = \{a\}$	K
C_6	1	2	3	2	3	1	$J_1 = K$; $Y_1 = \{e, a^5\}$	N-PC
C_7	1	2	3	3	1	2	$J_1 = K$; $Y_1 = \{e, a^3\}$	H_2
C_8	1	2	3	1	2	3	$J_1 = K$; $Y_1 = \{e, a\}$	N-PC

Table 1

From Lemma 4, *the perfect colorings turn out to be colorings using left cosets of a subgroup S of G,* $K \leq S \leq G$. *As seen in Table* 1 *there are four perfect colorings,* C_1, C_2, C_5 *and* C_7*(The corresponding S for each coloring is given in the last column). The remaining four colorings,* C_3, C_4, C_6 *and* C_8 *are non-perfect*(N-PC). *Let us consider* C_3, *which is associated with the partition P of G,* $P = P_1 \cup P_2 \cup P_3 \cup P_4$ *where* $P_1 = K \cup Ka^2 \cup Ka^4$, $P_2 = Ka$, $P_3 = Ka^3$*and* $P_4 = Ka^5$. *Also consider* C_4, *which is associated with the partition* $Q = Q_1 \cup Q_2 \cup Q_3 \cup Q_4$, *where* $Q_1 = K$, $Q_2 = Ka^2, Q_3 = Ka^4$ *and* $Q_4 = Ka \cup Ka^3 \cup Ka^5$. *Now under the element*
$a \in G, a(P_1 \cup P_2 \cup P_3 \cup P_4) = aP_1 \cup aP_2 \cup aP_3 \cup aP_4 = a(K \cup Ka^2 \cup Ka^4) \cup a(Ka) \cup a(Ka^3) \cup a(Ka^5) = (Ka \cup Ka^3 \cup Ka^5) \cup Ka^2 \cup Ka^4 \cup K = Q_4 \cup Q_2 \cup Q_3 \cup Q_1$ *or* $aP = Q$. *Thus by Definition* 7, C_3 *and* C_4 *are equivalent colorings. We can also verify that colorings* C_6 *and* C_8 *equivalent.* ■

Theorem 9. *Let G be a plane crystallographic group and* $H, K \leq G$ *where K is normal in G. Let* $G = \bigcup_{i \in I} \bigcup_{h \in H} hJ_iY_i$ *be a* $(Y_i, J_i) - H$ *coloring satisfying Theorem* 6.*There are six perfect colorings and two non-perfect such colorings that arise if G/K is the dihedral group of order* 6 *denoted by* D_3 *and* $[G : H] = 2$. *Moreover, both non-perfect colorings are equivalent.*

Proof. *Let* $G/K = \{K, aK, a^2K, bK, abK, a^2bK\}$ *be the dihedral group* D_3 *of order* 6. *The proper subgroups of G may be described as* $H_1 = K \cup aK \cup a^2K$ *and* $H_2 = K \cup bK$, $H_3 = K \cup abK$ *and* $H_4 = K \cup a^2bK$. *Since* $[G : H] = 2$ *we let* $H = H_1$.*Then the H-orbits are* $\{K, Ka, Ka^2\}$ *and* $\{Kb, Kab, Ka^2b\}$.

Using Theorem 6, we obtain the following color table where the colors 1,2,..., 6 are given to the right cosets of K in G.

	H			Hb				
	K	Ka	Ka^2	Kb	Kab	Ka^2b	J_i and Y_i used	
C_1	1	1	1	1	1	1	$J_1 = H$; $Y_1 = \{e,b\}$	G
C_2	1	1	1	2	2	2	$J_1 = H; J_2 = H$; $Y_1 = \{e\}; Y_2 = \{b\}$	H_1
C_3	1	1	1	2	3	4	$J_1 = H; J_2 = K$; $Y_1 = \{e\}; Y_2 = \{b\}$	N-PC
C_4	1	2	3	4	4	4	$J_1 = K; J_2 = H$; $Y_1 = \{e\}; Y_2 = \{b\}$	N-PC
C_5	1	2	3	4	5	6	$J_1 = K; J_2 = K$; $Y_1 = \{e\}; Y_2 = \{b\}$	K
C_6	1	2	3	2	3	1	$J_1 = K$; $Y_1 = \{e,a^2b\}$	H_4
C_7	1	2	3	3	1	2	$J_1 = K$; $Y_1 = \{e,ab\}$	H_3
C_8	1	2	3	1	2	3	$J_1 = K$; $Y_1 = \{e,b\}$	H_2

Table 2

There are six perfect colorings C_1, C_2, C_5, C_6, C_7 and C_8 corresponding to each of the subgroups S of G, $K \leq S \leq G$. The two non-perfect colorings, C_3 and C_4 are equivalent under the element $b \in G$. ■

Remark 10. From Theorems 8 and 9 given above we see that although the number of colorings listed are the same (since $[G : K] = 6$ and H/K is Z_3 for both cases), the number of perfect/non-perfect colorings vary because the quotient group G/K given in Theorem 8 is cyclic while that in Theorem 9 is dihedral.

We summarize the remaining results of our construction in Table 3. The proofs are omitted and can be patterned after that of Theorems 8and 9 above. The color tables for each case can also be constructed by means of Theorem 6. The notation in the table below are as follows: by PC we mean perfect colorings, N-PC are non-perfect colorings, $Z_j$we mean the cyclic group of order j, D_k the dihedral group of order $2k$, where j,k are integers and E is the trivial group containing the identity.

	G/K	H/K	$[G:H]$	PC	N-PC	Equivalent Non-perfect
1	Z_2	E	2	2	–	–
2	Z_3	E	3	2	3	all 3
3	Z_4	Z_2	2	3	4	occur in pairs
4	Z_4	E	4	3	11	4 pairs,last 3
5	Z_6	Z_3	2	4	4	occur in pairs
6	Z_6	Z_2	3	4	27	occur in 3's
7	D_3	Z_3	2	6	2	occur in pairs
8	D_3	Z_2	3	6	25	none
9	Z_8	Z_4	2	4	12	occur in pairs
10	Z_8	Z_2	4	4	158	occur in 4's,last 2
11	D_4	Z_4	2	10	6	occur in pairs
12	D_4	$Klein-4$	2	10	26	occur in pairs
13	D_4	Z_2	4	10	152	occur in 4's
14	Z_9	Z_3	3	3	39	occur in 3's
15	Z_{10}	Z_5	2	4	4	occur in pairs
16	Z_{12}	Z_6	2	6	22	occur in pairs
17	Z_{12}	Z_4	3	6	78	occur in 3's
18	Z_{12}	Z_3	4	6	262	occur in 4's, last2
19	D_6	Z_6	2	16	12	occur in pairs
20	D_6	D_3	2	16	38	occur in pairs
21	D_6	$Klein-4$	3	16	303	none
22	D_6	Z_3	4	16	252	occur in 4's

Table 3

We observe that the number of perfect/non-perfect colorings obtained varies depending not only on the group structure of G/K but also on that of H/K as well.

Example 11. *We now illustrate Theorem 9 by considering the uncolored pattern U given below whose symmetry group G is the plane crystallographic group p6m generated by* r, s, x, y.

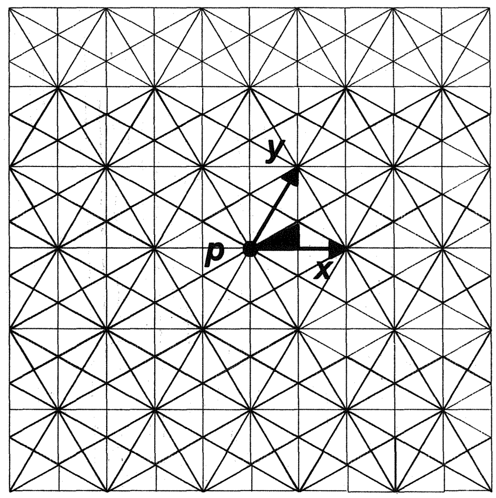


Figure 3: uncolored pattern U with symmetry group $p6m$

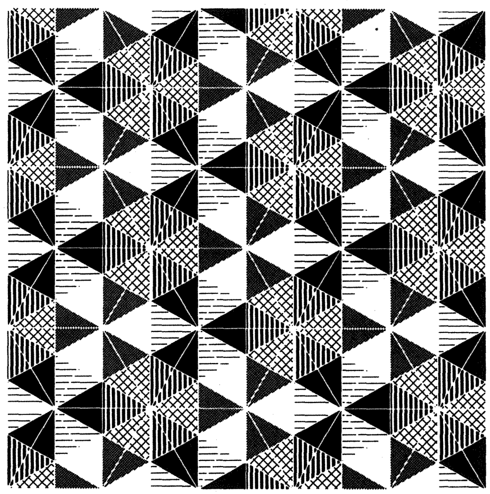
Let us choose the subgroups $H = \langle\, r^2, s, x, y \,\rangle$ *and* $K = \langle\, r^2, s, x^3, xy \,\rangle$ *of* G *which are plane crystallographic groups of types* $p31m$ *and* $p3m1$ *respectively where* $K \leq H \leq G$ *and* K *is normal in* G. *Note that* $[G : H] = 2$ *and* $[H : K] = 3$ *so that we can write* $G = H \cup Hr$, $H = K \cup Kx \cup Kx^2$ *or equivalently,* $G = (K \cup Kx \cup Kx^2) \cup (K \cup Kx \cup Kx^2)r$.

Let us first show how we obtain a particular coloring of U. *Suppose we consider* $J_1 = K$ and $J_2 = K$ *and we partition the set of right coset representatives of* H *in* G *into* $Y_1 = \{e\}$ *and* $Y_2 = \{r\}$. *We obtain the decomposition*

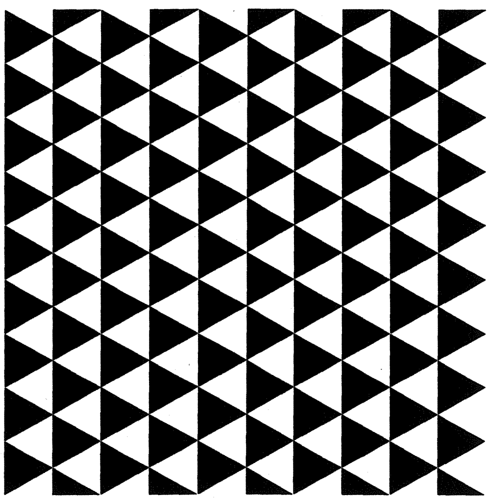
$$G = \bigcup_{i\in I} \bigcup_{h\in H} hJ_iY_i = \bigcup_{h\in H} h(K \cup Kr) = (K \cup Kr) \cup x(K \cup Kr) \cup x^2(K \cup Kr) = K \cup Kx \cup Kx^2 \cup Kr \cup Kxr \cup Kx^2r$$

which results in a coloring where all right cosets of K in G *are given different colors. If we assign the colors* 1, 2, 3,...,6 *to* K, Kx, Kx^2, Kr, Kxr *and* Kx^2r *respectively, we obtain the first colored pattern in Figure* 4. *This is a perfect coloring and is the same coloring referred to as* C_5 *in Table* 2. *Note that to generate the coloring we consider the triangular region colored black in Figure* 3 *the identity* e.

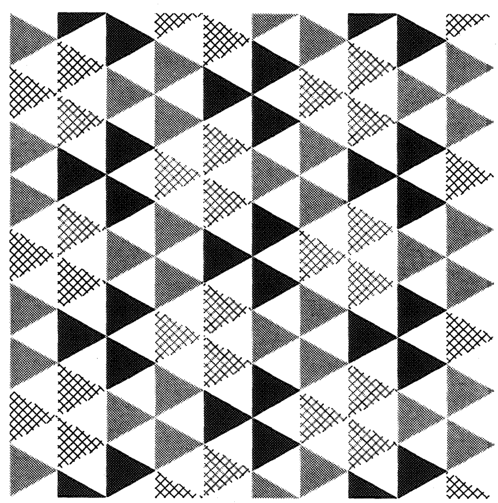
We also give in Figure 4 *the remaining six* $(Y_i, J_i) - H$ *colorings of* U *corresponding to* C_2, C_3, C_4, C_6, C_7 and C_8 *in Table* 2. C_2, C_6, C_7 and C_8 *are also perfect while* C_3 *and* C_4 *are equivalent and non-perfect. Notice that the* 60° *rotation* r *does not permute the colors in* C_3 *and* C_4 *so that these colorings are indeed non-perfect. Moreover, if we apply the rotation* r *to coloring* C_3, *we get coloring* C_4. *In these sense, colorings* C_3 *and* C_4 *are equivalent. In the actual colorings, the following shades were used to represent the color numbers* 1,2,3,...,6 *in Table* 2: 1 - *white,* 2-*black,* 3-*matte,* 4-*grey* , 5-*horizontal stripes* *and* 6-*vertical stripes.*



C5

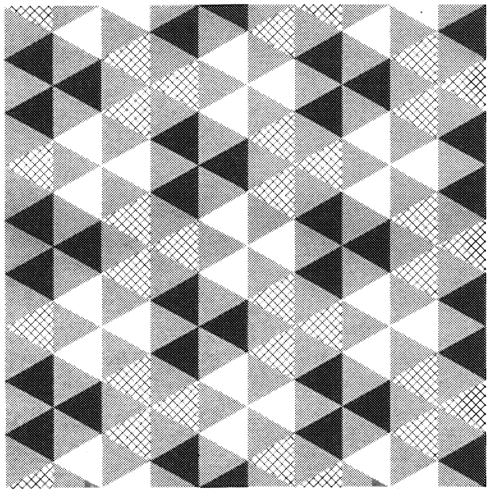


C2

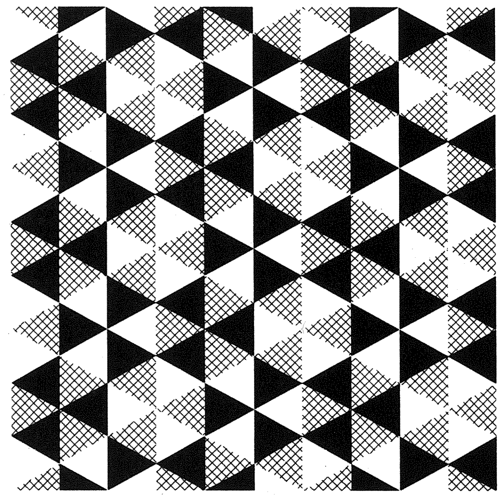


C3

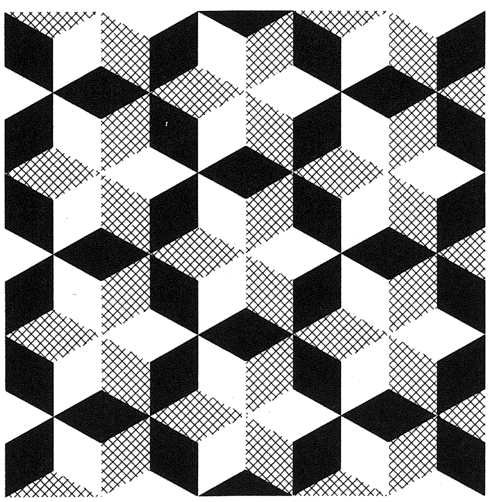
Figure 4: $(Y_i, J_i) - \langle r^2, s, x, y \rangle$ **colorings of** U



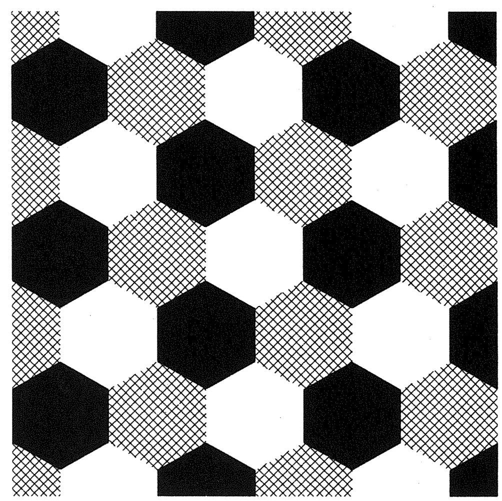
C4



C6



C7



C8

Figure 4: $(Y_i, J_i) - \langle r^2, s, x, y\rangle$ **colorings of** U **(cont.)**

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