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Mathematical Connections
in Art, Music, and Science

Beyond the Golden Section – the Golden tip of the iceberg

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Abstract

The Golden Section is considered by many as the pinnacle of perfect proportion. There are many other proportions which have unusual, interesting or equally valid properties which do not seem to have been studied. They do not *appear* to have been used, but this may be because they have not been documented or because no-one has deemed to look, or because they cannot be constructed using ruler and compasses. This paper is an attempt to look at more than the tip of the iceberg in terms of proportion.

Introduction

The Golden Section is perhaps the most commonly cited proportion both in art and architecture. I believe that there are many more proportions that have just as interesting properties and that the Golden Section is just the tip of the iceberg, albeit a special one. In describing other proportions and proportional systems, I hope that eyes can be opened and possibilities explored for their use, as well as preparing the way for exploration of their existence. It is well known that the Golden Section is found everywhere because it is sought after and, in many cases, found by careful selection of data to fit aspirations rather than fact. There are a multitude of exciting possibilities for other systems of proportions which may have been used in the past but, because no one has looked for them, their use is not known. Non-mathematicians find ways around constructions that are not theoretically possible with ruler and compasses and, with the advent of computers, more is possible. Because there is so much to explore, some proportions will only be introduced rather than explored in depth. There is enough to fill a book rather than these few pages.

The Golden Section

The Golden Section is often quoted as being the perfect proportion. There are many more myths and fallacies about it than there are truths [1,16]. Some erroneous cases like the Nautilus shell [2] are perpetuated when mathematicians do not check the facts [3, 4]. The Golden Section is often found simply because it is looked for, and the facts "adjusted" to fit the hypothesis of the searcher. The truth is somewhere in the middle ground. This is not to say that it is not important and that new facts cannot be found which, strangely, are not well known. A few important and unusual constructions are as follows.

One of the most simple constructions for the Golden Section, is Odom's construction [5] with the circumcircle of an equilateral triangle which is shown in figure 1. Join the centres of two sides and produce them to the circle. Then AB/BC is in the Golden Section. This is easily proved using the intersecting chords property (Euclid's Elements XIII.8). I believe Odom found it from the properties of the icosahedron in which the Golden Section abounds. It is odd that such a simple construction was only discovered so recently.

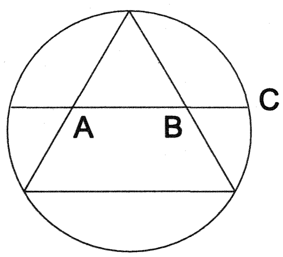


Figure 1

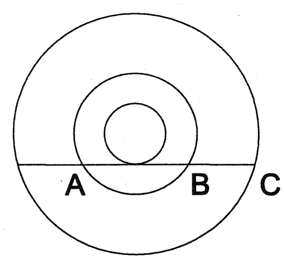


Figure 2

Sam Kutler's three circle problem shown in figure 2 is another simple construction, derived from it. The three circles are concentric; the middle circle has double the radius of the inner circle and half the radius of the outer circle and the line is tangent to the inner circle. Then AB/BC is in the Golden Section.

The Sacred Cut

This is the name given to the proportional construction shown in figure 3 by the Dane Tons Brunes [7].

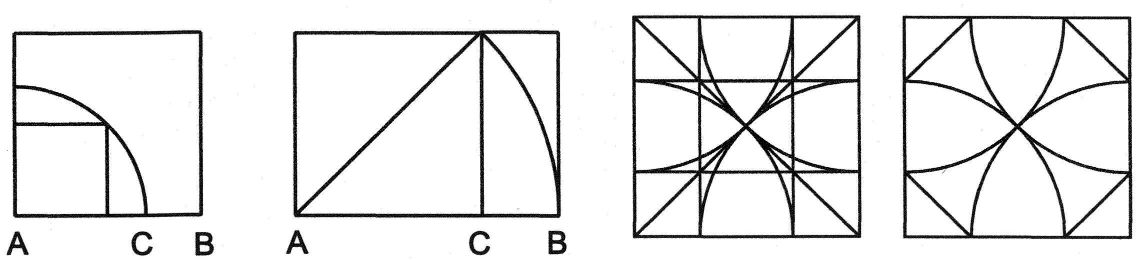


Figure 3

The two constructions shown on the left create C as the Sacred Cut of AB, giving AB/AC as $\sqrt{2}$:1. Brunes thought that ancient Greek references to the Golden Section are in fact to the Sacred Cut. The second construction has a similarity to the standard construction of the Golden Section (and the whirling squares configuration in figure 4) which uses the half diagonal of the square based on AC instead of the full diagonal. It may, in fact, be a more natural type of proportion for designs than the Golden Section since it is an easy and obvious construction. It is common in designs based on a square within a square, such as Tibetan Mandalas and is particularly common in Roman mosaics and has been found in the designs in the Laurentian Library [8] and in Roman House design and the layout of the city of Florence [9]. Paper sizes (the A series) in Europe are based on a proportion of $\sqrt{2}$:1 with A0 having an area of one square metre. Folding or cutting in half gives two pieces of the same proportion but rotated by 90°.

Dynamic symmetry and whirling squares

Although the Golden Section has been known since the time of the Ancient Greeks, and the Renaissance through Pacioli's book, it only came to real prominence at the end of the nineteenth century and the name has only been traced back to 1824 to a book by Martin Ohm (the brother of the discoverer of Ohm's law). The most famous manifestation is the whirling squares figure derived from a golden rectangle which allows the generation of an approximation to a logarithmic spiral using quadrants of circles (figure 4).

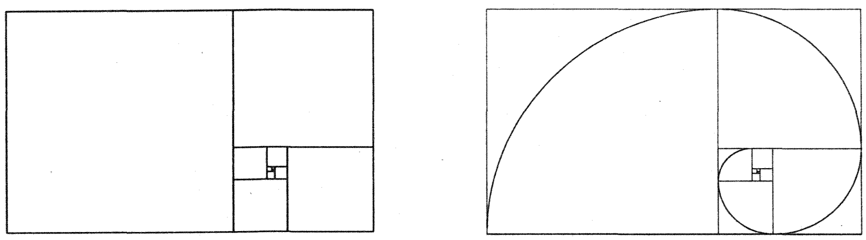


Figure 4

I have discussed the mathematics of the true logarithmic spiral fitting in the rectangle (which cannot go though the point where the square intersects the side of the rectangle without intersecting the rectangle) in another article [2]. This includes the construction of other spirals like the one shown in figure 5 which is complementary to the one in figure 4 since it uses *three* quadrants of the circle and which I have called the wobbly spiral. A number of these, combined as in the right of figure 5, show plant-like forms.

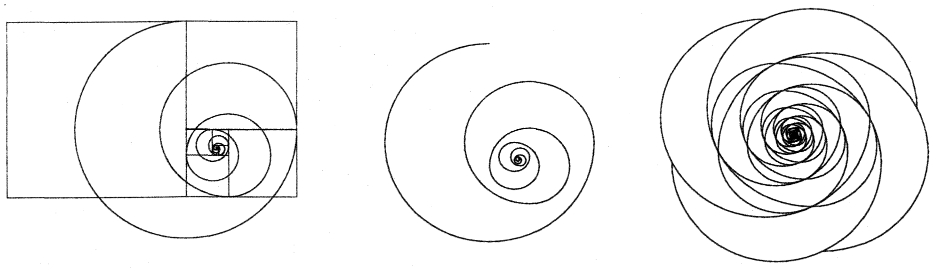


Figure 5

The equation of the wobbly spiral (not its approximation using arcs of circles) is:

$$r = (1 + 2k\sin(4\theta/3))ae^{\theta k}$$

where $k = \ln(\phi)/(3\pi/2)$ and ϕ is the Golden Section.

Such gnomonic constructions can be applied to any rectangle to yield other spirals. This is the system of Dynamic Symmetry of Jay Hambidge [10] and is achieved by constructing a perpendicular to a diagonal which gives rise to a similar rectangle as shown in figure 6, so that PBDQ has the same proportion as ABCD.

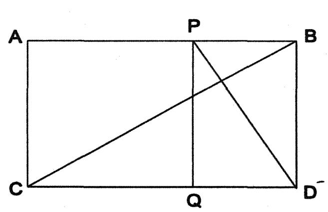


Figure 6

When the rectangle is a Golden Section one, the construction divides it into a square and another Golden Rectangle as shown here, but the system can be applied to any rectangle.

Dynamic symmetry has been very influential in the twentieth century both in art and architecture, with roots in the concept of gnomons. In looking below the tip of the iceberg, the concept can be developed in a number of ways, and as I will show later to study other systems of proportion. Edwards book on art deco design [11] is a good example of its use.

Squares from rectangles

Rather than take rectangles from rectangles, what happens if we take squares from rectangles other than the Golden Rectangle? In many cases another rectangle will result and the shape of the resulting rectangle obviously depends on the starting rectangle. If you continue to take a square off the resulting rectangle, in a similar manner to the whirling squares in figure 4, then in general the resulting rectangle will continually change shape. For some rectangles it may not be possible to remove a rectangle. There are also special cases. The most important one is the Golden Section where the resulting rectangle is the same shape as the original. A rectangle with the proportion 2:1 results in another square and you can go no further. This is the singularity, below which removal of a square yields another rectangle and above which it is not possible. One of the most interesting cases is the $\sqrt{2}$ rectangle which gives a rectangle from which two squares can be removed. Adding a square to the $\sqrt{2}$ rectangle gives a rectangle with the proportion $1+\sqrt{2}$ and the resulting diagram can be used to construct two spirals as shown in figure 7.

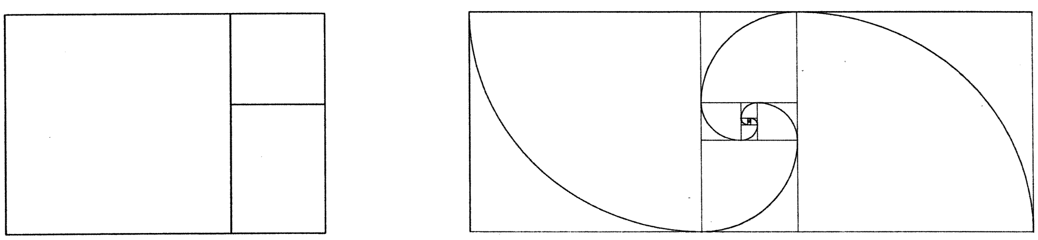


Figure 7

I can find no reference to the morphology of such systems. Many of the techniques used with Golden Section rectangles can be applied to the above proportional systems, but there is much more of the iceberg to investigate, so I will now move to some newer, unexplored possibilities.

The High-Phi division of a square

Martin Gardner, though retired, continues to write occasionally and I came across this division of a square with special properties in a student mathematics journal [12]. The puzzles originated with the New Zealand computer scientist Karl Scherer. The starting point was a puzzle to dissect a square into three similar shapes no two of which are congruent. There is an infinity of solutions, with examples in figure 8, where the slant line in the left part of this figure is inclined at various angles. Gardner thinks that the integer solution given may have the smallest values. The solution at the right is where the line is orthogonal to the sides. This cannot have integer solutions (although it can have approximate ones just like the Fibonacci numbers can be used to approximate to a Golden Section rectangle). If the lengths are marked as in the centre diagram, then x has interesting properties. (The right diagram has been reduced to a unit square.)

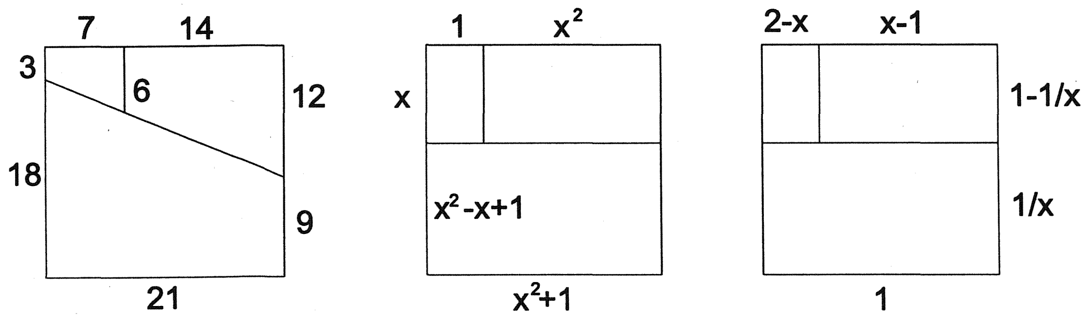


Figure 8

An interesting point about the left part of figure 8 is that the three areas are in the ratio $1^2:2^2:3^2$ and perimeters in the ratio 1:2:3.

The division yields three rectangles with the same proportions. It is a logical child of the division of the Golden Rectangle in Figure 4 and consequently Martin Gardner has given it the name High-Phi to match the often used symbol for the Golden Section which is the Greek letter phi (ϕ).

The equation for x is a cubic:

$$x^3 - 2x^2 + x - 1 = 0$$

which solves to the value 1.75487766624669276. The equation also gives rise to an associated recurrence sequence (like the Fibonacci sequence for the Golden Section) which is:

$$u_n = 2u_{n-1} - u_{n-2} + u_{n-3}$$

for which the first 20 terms of the sequence are 0, 1, 1, 1, 2, 4, 7, 12, 21, 37, 65, 114, 200, 351, 616, 1081, 1897, 3329, 5842, 10252, 17991, 31572, 55405, 97229, 170625, 299426, 525456, 922111, 1618192...

For lovers of Fibonacci curiosities, note how u_{29} is 1618192. The ratio of successive terms converge more slowly than the Fibonacci sequence, so that (if u_0 is 0) then u_{29}/u_{28} is 1.7548776665715...

The reason for Martin Gardner giving it the name High-Phi, apart from the general similarity of the diagram, is the relationship:

$$(x - 1)^2 = 1/x \quad \text{to compare with } x - 1 = 1/x \text{ for } \phi$$

There are other simple relationships derivable from this:

$$x = 1 + 1/\sqrt{x} \qquad \sqrt{x} - 1 = 1/x^2 \qquad x\sqrt{x} = x + 1/x \qquad 1 + 1/(x - 1) = x + 1/x$$

The areas of the three rectangles with the notation in the centre of figure 8 are x, x^3 and x^4 whereas in the one on the right they are $1/x^4$, $1/x^2$, $1/x$ and thus giving the relationship:

$$1/x + 1/x^2 + 1/x^4 = 1$$

and a secondary equation:

$$x^4 - x^3 - x^2 - 1 = 0$$

which gives another recurrence relationship for the sequence above:

$$u_n = u_{n-1} + u_{n-2} + u_{n-4}$$

This number 1.75487766624669276 is related to the Plastic Number (see below).

Super-High-Phi

Having got this far, begs the question of what is the proportion that occurs when division of a rectangle results in three rectangles which are the same shape as the starting one as shown in figure 9.

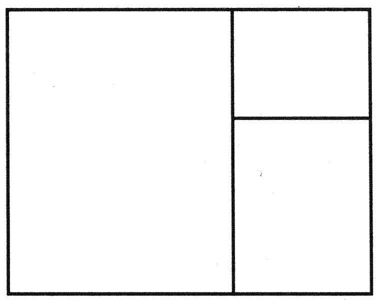


Figure 9

This leads us back to the Golden Section since the ratio of the sides of the rectangle is $\sqrt{\phi}$. The right hand side is divided in the Golden Section.

It is obviously difficult to avoid the Golden Section, but since I am trying to do so, I will move on.

The Tribonacci constant

The Golden Section is essentially two dimensional, although it is found in two of the Platonic solids, the dodecahedron and the icosahedron and in four dimensional polytopes, but not in higher dimensional ones. There is another pair of proportions hidden in the Archimedean solids. The diagonals of the snub cube and many other properties [13] are functions of the number 1.83928675521416… (which I have called η) which is the root of the equation

$$x^3 - x^2 - x - 1 = 0$$

which is the associated equation of the sequence: 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149 … that is the recurrence:

$$u_n = u_{n-1} + u_{n-2} + u_{n-3}$$

each number being the sum of the previous three terms rather than the previous two as is the case with the Golden Section which is often called the Tribonacci sequence.

The snub cube fits in the cube, with the square of the snub cube rotated on the face of the cube by an angle whose tangent is η.

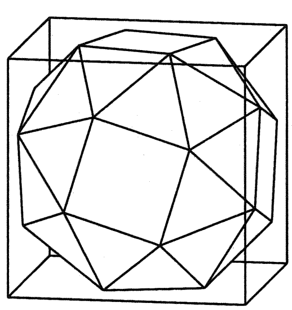


Figure 10

The snub dodecahedron also has a constant associated with it, the number 1.9431512592438817 which is the root of the equation:

$$x^3 - x^2 - x - \phi = 0$$

where ϕ is the Golden Section.

The heptagon and other polygons

Another approach is to consider regular polygons. Whereas the Golden Section is derived from the pentagon, other polygons offer numbers with much richer properties. They have not been studied in depth because of a quirk of the history of mathematics. Only a limited number of regular polygons can be constructed with the tools of the ancient Greek mathematicians, the straightedge and compasses. This is not to say that they cannot be constructed, but as the Golden Section obscures all others proportions, so construction of polygons only seems "valid" if carried out with the "approved" tools. Even the Greeks had other methods, for example using curves or, in the case of the heptagon using the so called "lost neusis" of Archimedes [14]

The pentagon only has one type of diagonal but the heptagon has two which I have labelled using Steinbach's symbols [15].

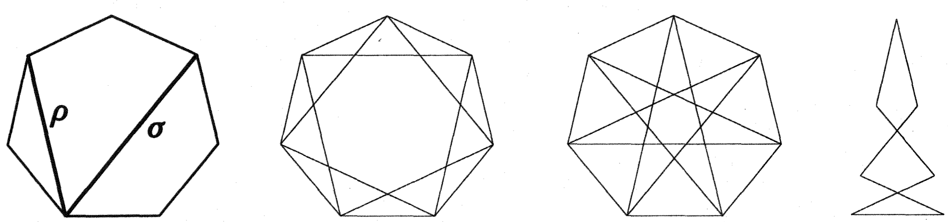


Figure 11

This allows two types of stellated polygons and the one at the right of figure 11 where the sides are all of the same length.

The two constants σ and ρ are roots of the equations:

$$\rho^3 - \rho^2 - 2\rho + 1 = 0 \quad \text{and} \quad \sigma^3 - 2\sigma^2 - \sigma + 1 = 0$$

giving three roots for each, but if the side of the heptagon is unity then $\sigma = 2.246979603717..$ and ρ is 1.8019377358048... with various relationships between the roots:

$$\frac{1}{\sigma} + \frac{1}{\rho} = 1 \qquad \frac{1}{\sigma} = \sigma - \rho \qquad \frac{\rho}{\sigma} = \rho - 1 \qquad \frac{\sigma}{\rho} = \sigma - 1$$

$$\rho\sigma = \rho + \sigma \qquad \sigma^2 = \rho + \sigma + 1 \qquad \rho^2 = 1 + \sigma$$

Which gives geometric interpretations like a σ rectangle and a ρ rectangle fitting together to give a square and a ρ removed from a σ rectangle resulting in another σ rectangle. There is scope for a whole series of logarithmic spirals.

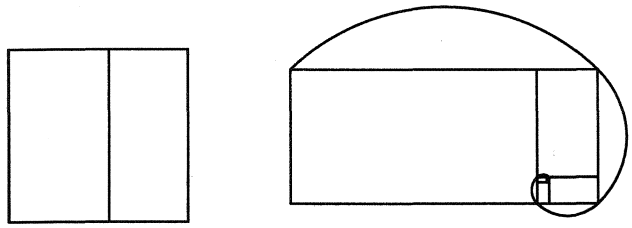
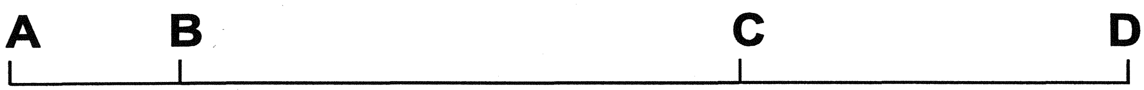


Figure 12

The familiar division of a line and fixing the position of the Golden mean also has its Heptagonal equivalent, but this time there are two points because we are dealing with a cubic construction not a quadratic one.



thus AB : CD : BC = CD : AC : BD = BC : BD : AD or writing these divisions in terms of σ and ρ, then $1 : \rho : \sigma \ = \ \rho : 1 + \sigma : \rho + \sigma \ = \ \sigma : \ \sigma + \rho : 1 + \rho + \sigma$ which I find more satisfying than the simple Golden Ratio perhaps because it has slightly more complexity.

From the equations for σ and ρ, there are recurrence relationships for the heptagonal equivalent of the Fibonacci numbers which I will term the σ-Heptanacci and ρ-Heptanacci numbers, namely the sequence:
0, 1, 1, 3, 6, 14, 31, 70, 157, 353, 793, 1782, 4004 ......
0, 1, 1, 3, 4, 9, 14, 28, 47, 89, 155, 286, 507, 924 ....

There is also scope for writing stories or puzzles to correspond with Fibonacci's rabbits. Consider the following. In the following rows of letters, the sequence in a row is obtained from the previous row by substitution. "A" becomes "C", "B" becomes "BC" and "C" becomes "ACB". The fractal nature of the rows is evident if you notice that each row is always the beginning of the next but one row.

A
C
ACB
CACBBC
ACBCACBBCBCACB
CACBBCACBCACBBCBCACBBCACBCACBBC
ACBCACBBCBCACBCACBBCACBCACBBC ......

If you count the number of each letter in a row, then for "A" you get 1, 0, 1, 1, 3, 6, 14, 31, 70, 157, 353, 793 .., and for "B" 0, 0 , 1, 2, 5, 11, 25, 56, 126, 283, 636, 1429 .... and for "C" 0, 1, 1, 3, 6, 14, 31, 70, 157, 353, 793, 1782.

The sequence for B arises from the expansion of the powers of σ and so there are other Heptanacci sequences also.

Another aspect is the types of triangles which have angles which are multiples of $\pi/7$ as denoted by the numbers in the following figure:

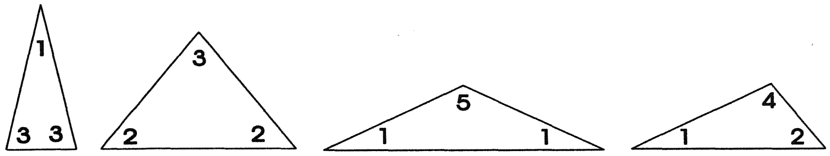


Figure 14

where the sides of triangle 133 are in the ratio $1 : \sigma : \sigma$, of 223 are $\sigma : \rho : \rho$, of 115 are $\rho : 1 : 1$ and of 124 (which incidentally are angles in geometric progression) are $1 : \rho : \sigma$. Such triangles in turn lead to the consideration of aperiodic tilings equivalent to the Penrose tilings as shown in figure 15.

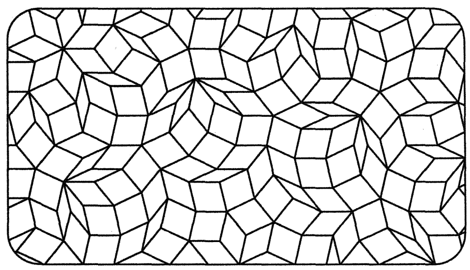


Figure 15

These are but a few aspects of the heptagonal possibilities. As one increases the sides of the polygon so different properties appear with the increasing number of diagonals. Steinach [15] deals with some of these properties, but they are even more unexplored than the heptagon.

The Plastic Number

This is the discovery of the Dutch architect Dom Hans van der Laan (1904-1991). He started out trying to follow the work of a fellow architect who used the Golden Section, but found he could not get it to work in three dimensions. His work and how he came to discover the Plastic Number is the subject of books and three articles by his champion, the English architect Richard Padovan [17, 18, 19]. A mention of his name with some confusion of information by Ian Stewart [20] has led to the recurrence sequence being associated with Padovan.

The Golden Section was considered essentially two dimensional by van der Laan, although it obviously does have three dimensional manifestations in the dodecahedron and icosahedron. He looked for an equivalent. It is worth bearing in mind that the equation for the Golden Section is a quadratic, so the Plastic Number is the root of a cubic, derived from the equation:

$$p^3 - p - 1 = 0$$

which has the real root 1.3247179572447… and equations which are reminiscent of Golden Section related ones:

$$p = \frac{1}{p} + \frac{1}{p^2} \qquad p^2 = \frac{1}{p} + 1 \qquad p^3 = p + 1 \qquad p^4 = p^5 - 1$$

The associated recurrence relation is:

$$u_n = u_{n-2} + u_{n-3}$$

and there also others:

$$u_n = u_{n-1} + u_{n-5} \qquad \text{and} \qquad u_n = u_{n-3} + u_{n-4} + u_{n-5}$$

with the associated sequence 1,1,1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114.....

The sequence for High-Phi above (0, 1, 1, 1, 2, 4, 7, 12, 21, 37, 65, 114,..) forms alternate numbers in the Plastic Number sequence. This is because the Plastic Number is the square root of High-Phi. Striking out the High-Phi numbers and looking at the remainder gives another sequence which also obeys the High-Phi recurrence.

Geometrically, the Plastic Number offers both two and three dimensional possibilities. The recurrence $u_n = u_{n-1} + u_{n-5}$ leads to a set of whirling triangles to match the whirling squares of the Golden Section. These are equilateral triangles.

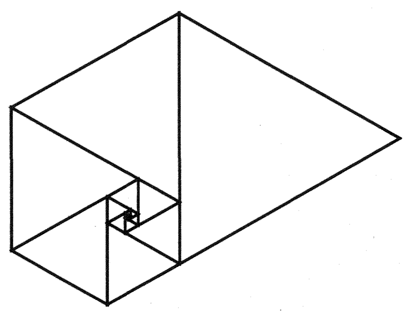


Figure 16

which in turn leads to a spiral which has special properties:

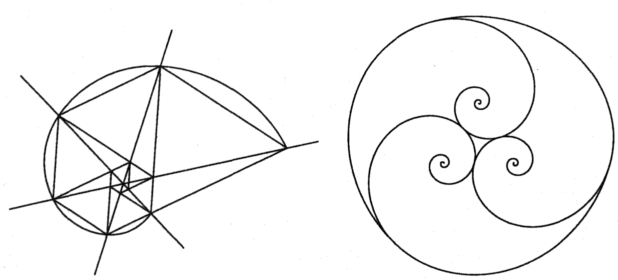


Figure 17

The spiral was discovered many years ago by Rutherford Boyd [21] when he was looking for spirals which touched a number of times. In this case, join the poles of the spirals and the lines go through the touching points.

In three dimensions, a "whirling" takes place with rectangular boxes leading to a helical spiralling of the boxes. Figure 18 shows how the boxes build up to the Plastic cuboid, using the Plastic sequence. The cuboids have dimensions 1,1,1 then 1,1,2 then 1,2,2 then 2,2,3 then 2,3,4 then 3,4,5 then 4,5,7 and so on.

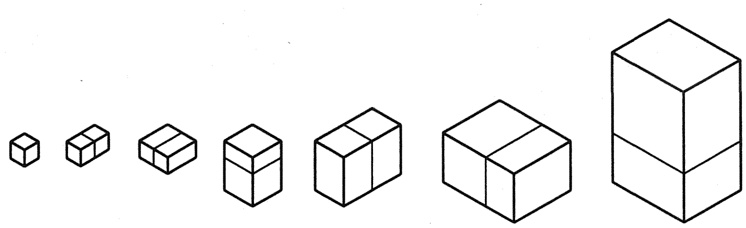


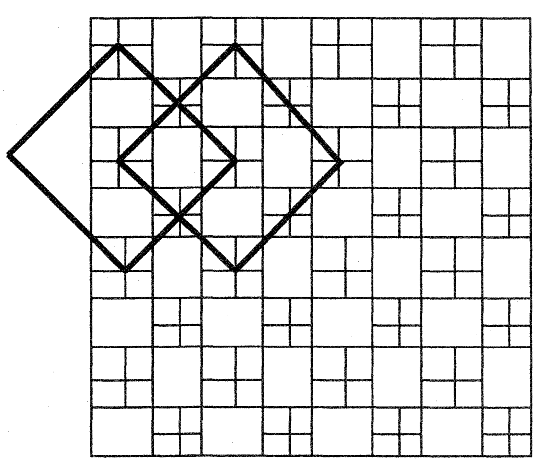
Figure 18

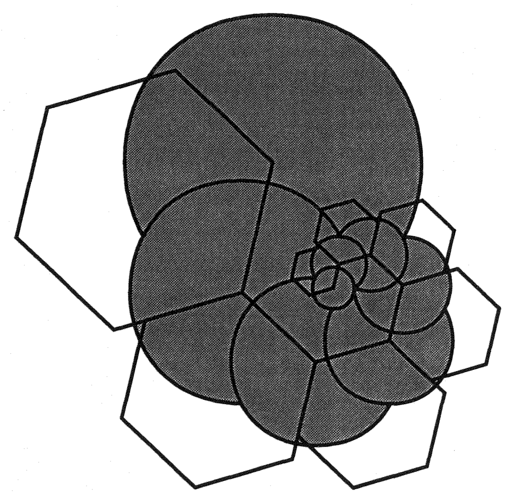
This sequence has been drawn in isometric projection, which is easier to draw but does not show the full beauty of the object.

In this approximation to the true Plastic cuboid, the addition each time results in a square added to one face. This can be built up to create a three dimensional spiral by combining quarter circles on the square faces of the boxes in the same way as a spiral is built in the Golden Section rectangle in figure 4.

From Theory to Practice

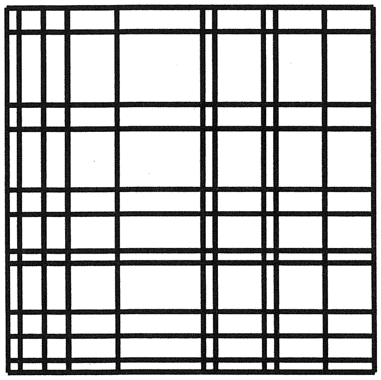
The above ideas are starting points, both in rich areas of mathematics but also as inspiration for art. Considering how artists have used the Golden Section by being inspired by the myths which arose in the nineteenth century, then there are many lifetimes' work here. This needs the computer in many cases since a number of these ratios cannot be constructed with standard ruler and compasses. However, square roots can and the approximations using recurrence sequences which are similar to the use many artists make of the Fibonacci sequence require no special tools. If anything, the possibilities are overwhelming. I have many examples in sketchbooks which have not gone further because too many other possibilities suggest themselves. They are the geometrical equivalent of equivalent of the Chinese curse of "may you live in interesting times". Such possibilities are easier to show in a presentation, so this paper is a theoretical background. The following examples are a taster.

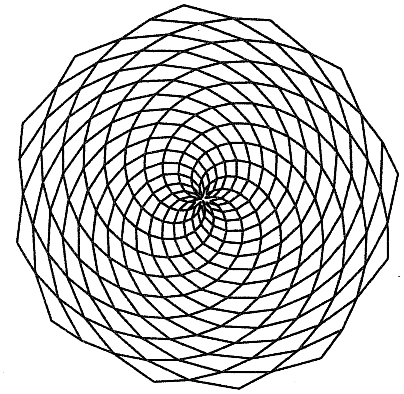




The first of the two above was designed using the heptagonal division of a square. The second comes from the whirling triangles using the Plastic number.

The mathematician Lagrange was probably the first to prove that if you take the Fibonacci sequence, and reduce it, for example, MOD 4, then you get a periodic sequence 0,1,1,2,3 … Now other sequences subjected to this transformation give seemingly random patterns. The next example shows how such patterns can be used to generate grids which, in this case might be used to create aperiodic rectangular tilings reminiscent of Mondrian. They could be also used for weaving pseudo-tartan patterns. This one uses Heptanacci numbers.





Finally, the Fibonacci numbers are not the only ones to give sunflower-like designs. This is set of spirals using the numbers 12 and 17 which are found in the continued fraction of the square root of 2 which is also used to define the special angle of the spiral.

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