BRIDGES
Mathematical Connections in Art, Music, and Science

Symbolic Logic with a Light Touch

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Abstract

In this paper, we discuss a unit on Symbolic Logic which has been designed in the context of a course entitled "Mathematics from a Humanist Perspective." The challenge of such a unit is that it must keep a light tone, avoid the use of heavy deductive machinery, and have relevance in the eyes of students. The objective of the unit is to bring about an understanding of the process of formal reasoning by using deductive rules which are elementary and well motivated. This paper contains two innovations: First, we have devised a deductive system which is very easy to use. A second innovative feature is the introduction of natural-language logic puzzles whose translation into symbols is quite straightforward, and whose solution by symbolic processing is easier to carry out than a solution by verbal reasoning. This last fact is especially useful in demonstrating the value of formal reasoning to students.

For about ten years I have been teaching a course entitled "Mathematics from a Humanist Perspective", which is open to non-science students at Bucknell University. Over the years I have experimented with different mixes of topics and a variety of course formats. Despite my familiarity with the demands of the course and the profile of my students, it is absolutely a surprise, every time, to discover what topics ignite the interest of students and what other topics leave them cold. The successful topic may be taught again the following year to see if the effect persists. And if it does persist --- if five or six successive student populations learn the topic and love it --- one yields to the conclusion that for reasons which transcend understanding, this topic "works": It is an authentic connection between the mathematical imagination and the open curiosity of active learners. It is a kind of conducting rod between mathematics and the arts.

One of these surprises was the fact that students from every walk of the academic spectrum could be made to enjoy learning symbolic logic. Admittedly, it was a somewhat unconventional take on symbolic logic, but it was neither watered down nor was its rigor diminished in any way. The Unit that I designed was confined to the propositional calculus; but the approach lends itself easily to various extensions, in particular to the addition of individual variables and quantifiers.

Few things are more important today, for students who must understand the key ideas --- the truly generative ideas --- of the ambient scientific culture, than to grasp the difference between empirical knowledge and knowledge attained by deduction. Generally, empirical knowledge is clear. But the process of formal reasoning --- the fact that it begins with unproved assumptions, and uses mechanical rules to derive conclusions from the assumptions --- that it does not produce any "new" knowledge except what is implicit in the premises, somehow these facts are not widely understood. The purpose of a Unit on Logic is to impart this understanding and make it fully explicit. It is our belief that this understanding is fundamental for relating mathematics to the humanities. Moreover, it is a big step toward clarifying the sense in which computers are able to generate music and art --- by starting with explicit 'aesthetic premises' and procedural rules and using these to build formal constructions blindly and mechanically.

The Unit is built around a set of logical puzzles which were originally designed to be solved by insight and deft reasoning, but which may also be solved symbolically by rigorous use of logical rules. Such problem sets have been around for a long time, and in most mathematics libraries you may find dusty volumes of logic puzzles, some published before the turn of the century; surprisingly you will notice a considerable overlap in the contents. But without a doubt the most charming and inventive re-creation of logic puzzles may be found in the books of Raymond Smullyan, which shine with new relevance. A selection of these puzzles, quoted from Raymond Smullyan, will be used in this article. I will begin with an example:

"THE ISLAND OF KNIGHTS AND KNAVES. *There is a wide variety of puzzles about an island in which certain inhabitants called 'knights' always tell the truth, and others called 'knaves' always lie. It is assumed that every inhabitant of the island is either a knight or a knave.*
Problem: In this problem, there are only two people, **a** and **b**, each of whom is either a knight or a knave; **a** makes the following statement: 'At least one of us is a knave'.
What are **a** and **b** ?" (Smullyan [1]).

In order to solve the problem symbolically, each statement in the problem must be represented by a distinct letter. For example, "***a** is a knight*" is symbolized by the letter A. Therefore, "**a** is *not* a knight", in other words "***a** is a knave*", is symbolized by $\neg A$ (not-A). Similarly, "***b** is a knight*" is symbolized by B, hence "***b** is a knave*" is $\neg B$. So for example,

"**a** is a knight and **b** is a knight" is symbolized by $A \wedge B$.
"if **a** is a knight, then **b** is a knight" is symbolized by $A \Rightarrow B$.
"**a** is a knight, or **b** is a knave" is symbolized by $A \vee \neg B$.

In our problem, the statement which **a** makes is symbolized as follows : $\neg A \vee \neg B$. (This is the symbolic form of "not-A or not-B"). Of course, the statement is not necessarily true. It is true if **a** happens to be a knight. But if **a** happens to be a knave, the statement is false, that is, its negation is true. The negation of $\neg A \vee \neg B$ is $A \wedge B$. So, in order to express this problem in symbols, we must write the fact that: *if **a** is a knight,* then $\neg A \vee \neg B$ is true. And *if **a** is a knave*, then the opposite is true, that is, $A \wedge B$ is true. These two facts are the premisses of this problem: They are the facts given at the start, with which we must work to deduce the solution.

Premisses: (P1) $A \Rightarrow (\neg A \vee \neg B)$

(P2) $\neg A \Rightarrow (A \wedge B)$

Solution:

	Reason
1. Assume $\neg A$.	
2. $\therefore$ $A \wedge B$.	From premiss (P2) and rule MP (modus ponens).
3. $\therefore$ A	Step 2 and Rule CON (rule of conjunction).
4. *Contradiction with Step 1.*	
$\therefore$ $\neg A$ cannot be true.	
$\therefore$ A	
5. $\therefore$ $\neg A \vee \neg B$	Step (4), premiss (P1) and rule MP.
6. $\therefore$ $\neg B$	Steps (4) and (5), and Rule MP' (see next page).

Our conclusions are: A, and $\neg B$. In other words, **a** is a knight, and **b** is a knave.

Note that the suggested solution is organized like a proof of elementary geometry. But it is more abstract, because the lines of the solution are entirely symbolic, and the rules of deduction are entirely mechanical. What is most surprising, perhaps, is how little formal logic is needed to solve these problems: Students make use of four logical equivalences, and four rules of deduction. For a wide range of problems, this small amount of machinery is sufficient to write formal proofs which are completely rigorous and have no logical gaps.

I generally begin the Unit with no more than a two-page summary of formal logic, which is included here as the Appendix to this article. The students learn the four basic logical connectives and their truth-tables. Then four fundamental equivalences are explained:

1. $\neg(A \wedge B) \equiv \neg A \vee \neg B$
2. $\neg(A \vee B) \equiv \neg A \wedge \neg B$
3. $\neg(A \rightarrow B) \equiv A \wedge \neg B$
4. $(A \rightarrow B) \equiv (\neg B \rightarrow \neg A)$

Finally, logical deduction is presented as an essentially mechanical procedure carried out on logical sentences by using the following four rules of inference:

From A and $A \rightarrow B$, deduce B.	*(Rule MP)*
From A and $\neg A \vee B$, deduce B.	*(Rule MP')*
From $A \wedge B$, deduce A. (Also, deduce B)	*(Rule CON)*
From A together with B, deduce $A \wedge B$.	*(Rule CON')*

Using nothing more than this simple machinery, many types of problems --- at many levels of difficulty --- can be solved in precisely the same manner as the example given above.

The strength of our approach is the simplicity of the deductive apparatus that is required. It is widely acknowledged that the limiting factor in teaching deductive or symbolic logic to students outside the sciences is the daunting complexity of the formalism, and the large number of rules of inference in the standard texts on logic. What is presented here is an unconventional logical system – a combination of four logical equivalences with four rules of inference. Such a system can be learned quickly and is very easy to use. Here is another, somewhat more difficult problem, together with its solution.

LOVE AND LOGIC.

In the problem which follows, we turn from the logic of chivalry to the logic of love. Note that these problems do not take place on the island of knights and knaves. Thus, the protagonists of the following problem are neither knights nor knaves---just folks like you and me.

Problem. I know three girls, called Marcia, Sue and Dianne, and my heart is a-flutter. My feelings for these girls may be summed up as follows:

1. I love at least one of the three girls.
2. If I love Sue but not Dianne, then I also love Marcia.
3. I either love both Dianne and Marcia, or I love neither one.
4. If I love Dianne, then I also love Sue.

Which of the girls do I love? (From Smullyan [1]).

To solve this problem symbolically, the letter M represents the proposition "I love Marcia", S stands for "I love Sue", and D represents "I love Dianne". Thus, $\neg M$ stands for "I don't love Marcia", and so on. The four premisses are symbolized as follows:

(P1) $S \vee M \vee D$

(P2) $(S \wedge \neg D) \Rightarrow M$

(P3) $(D \wedge M) \vee (\neg D \wedge \neg M)$

(P4) $D \Rightarrow S$

Solution.

	Reasons
1. Assume $\neg(D \wedge M)$.	
2. $\therefore$ $\neg D \wedge \neg M$	Step 1, premiss (P3) and rule (MP')
3. $\therefore$ $\neg D$ as well as $\neg M$	Step 2, rule (CON)
4. $S \vee M$	Premiss (P1), step 3, rule (MP')
5. S	Steps 4 and 3, rule (MP')
6. $S \wedge \neg D$	Steps 3 and 5, rule (CON')
7. M	Premiss (P2), step 6, (MP)
8. Contradiction between Steps 3 and 7. Thus, $\neg(D \wedge M)$ cannot be true. $\therefore$ $D \wedge M$	
9. D as well as M	Step 8, Rule (CON)
10. S	Premiss (P4), step 9, rule (MP).

Our conclusions are D, M and S. As I feared, I love all three girls.

The following problem is one of my favorites.

"FROM THE FILES OF INSPECTOR CRAIG

Inspector Leslie Craig of Scotland Yard has kindly consented to release some of his case histories for the benefit of those interested in the application of logic to the solution of crimes.

Problem. An enormous amount of loot had been stolen from a store. The criminal (or criminals) took the heist away in a car. Three well-known criminals, **a**, **b** and **c**, were brought to Scotland Yard for questioning. The following facts were ascertained:

(1) No one other than **a**, **b** and **c** was involved in the robbery.

(2) **c** never pulls a job without using **a** (and possibly others) as an accomplice.

(3) **b** does not know how to drive.

Is **a** innocent or guilty?" (Smullyan [1]).

To solve this problem in symbols, the letter A is used for the proposition "**a** is guilty", so that $\neg A$ represents "**a** is innocent". Likewise for the other protagonists. The premisses of this problem are as follows:

(P1) $A \vee B \vee C$

(P2) $C \Rightarrow A$

(P3) $B \Rightarrow A \vee C$

Solution	Reasons
1. Assume $\neg A$	
2. $B \vee C$	Step 1, premiss (P1), rule (MP')
3. $\neg A \Rightarrow \neg C$	Premiss (P2), equivalence 4
4. $\neg C$	Steps 1 and 3, rule (MP)
5. B	Steps 2 and 4, rule (MP')
6. $A \vee C$	Step 5, premiss (P3), rule (MP)
7. C	Steps 1 and 6, (MP')
8. Contradiction, steps 4 and 7.	
Thus, $\neg A$ cannot be true.	
$\therefore A$	

The variety of problems like the ones given above, whose solution can be found by "computation", is virtually endless. The intricacy and distinctiveness of problems can be increased if the logical system is extended to allow quantifiers. In that case, two additional equivalences are needed:

$$5.\quad \neg(\exists x)P(x) \equiv (\forall x)[\neg P(x)]$$
$$6.\quad \neg(\forall x)P(x) \equiv (\exists x)[\neg P(x)]$$

The last problem presented here involves the use of quantifiers. It is included here for the perusal of the hardy reader.

"The Asylum of Doctor Tarr and Professor Fether." (Adapted from Smullyan [2]).
Inspector Craig of Scotland Yard was called over to France to investigate an insane asylum where it was suspected that something was wrong. Each inhabitant of the asylum, patient or doctor, was either sane or insane. Moreover, the sane ones were totally sane and a hundred percent accurate in all their beliefs. The insane ones were totally inaccurate in their beliefs. Everything true they believed to be false, and everything false they believed to be true.

It was known that this asylum contained either a sane patient or an insane doctor. So Inspector Craig interviewed Doctor Tarr and Professor Fether in the following words:

Craig: Tell me, Doctor Tarr, are all the doctors in this asylum sane?

Tarr: Of course they are!

Craig: What about the patients? Are they all insane?

Tarr: At least one of them is.

Craig (to Professor Fether): Dr. Tarr said that at least one patient here is insane. Is that true?

Fether: Of course it is true! All the patients in this asylum are insane.

Craig: What about the doctors? Are they all sane?

Fether: At least one of them is.

Craig: What about Dr. Tarr? Is he sane?

Fether: Of course he is! How dare you ask me such a question?

At this point Craig realized the full horror of the situation. What is it?

The statements may be translated into symbols as follows: If x is any inhabitant of the asylum, D(x) is the assertion that x is a doctor, hence $\neg D(x)$ asserts that x is a patient. Also, S(x) asserts that x is sane, so $\neg S(x)$ asserts that x is insane. The letter t stands for Tarr and f stands for Fether. The fact that the asylum has either a sane patient or an insane doctor is conveyed in Premiss (P1), and the remaining premisses are the two doctors' statements.
(Recall that if x is not sane, all his assertions are false. This is used in (P4) and (P7)).

(P1) $(\exists x)[\neg D(x) \wedge S(x)] \vee (\exists x)[D(x) \wedge \neg S(x)]$

(P2) $S(t) \Rightarrow (\forall x)[D(x) \Rightarrow S(x)]$

(P3) $S(t) \Rightarrow (\exists x)[\neg D(x) \wedge \neg S(x)]$

(P4) $\neg S(t) \Rightarrow (\forall x)[\neg D(x) \Rightarrow S(x)]$

(P5) $S(f) \Rightarrow (\forall x)[\neg D(x) \Rightarrow \neg S(x)]$

(P6) $S(f) \Rightarrow (\exists x)[D(x) \wedge S(x)]$

(P7) $\neg S(f) \Rightarrow (\forall x)[D(x) \Rightarrow \neg S(x)]$

(P8) $S(f) \Leftrightarrow S(t)$

Solution	Reason
1. Assume S(f)	
2. Assume S(t)	
3. $(\forall x)[D(x) \Rightarrow S(x)]$	Step 2, premiss (P2), rule (MP)
4. $\neg(\exists x)[D(x) \wedge \neg S(x)]$	Step 3, equivalences 5 and 3.
5. $(\exists x)[\neg D(x) \wedge S(x)]$	Step 4, premise (P1), rule (MP')
6. $(\forall x)[\neg D(x) \Rightarrow \neg S(x)]$	Step 1, premiss (P5), rule (MP)
7. $\neg(\exists x)[\neg D(x) \wedge S(x)]$	Step 6, equivalences 5 and 3.

8. Contradiction, steps 5 and 7.
Thus, S(t) cannot be true.
$\therefore$ $\neg$S(t) *(On condition of step 1)*
9. $\therefore$ $\neg$S(f) *(On condition of step 1)* Step 8, premiss (P8), rule (MP)
10. Contradiction, steps 1 and 9.
Thus, S(f) cannot be true.
$\therefore$ $\neg$S(f)
11. $\therefore$ $\neg$S(t) Step 10, premiss (P8), rule (MP)
12. $(\forall x)[\neg D(x) \Rightarrow S(x)]$ Step 11, premiss (P4), rule (MP)
13. $(\forall x)[D(x) \Rightarrow \neg S(x)]$ Step 10, premiss (P7), rule (MP)

Thus, it turns out that all the patients are sane and all the doctors are insane.

In the course of working these problems, students learn three important skills, and the insights which come with these skills:

1. Students come to recognize that, in certain limited domains of the English language, sentences are made up of a small number of fundamental propositions connected together by means of logical connectives.
2. Once the structure of these sentences is understood (a skill which the students learn), the sentences may be written in fully symbolic form. In this symbolic representation the structure of the sentences is transparent.
3. Finally, students learn that deduction is essentially a mechanical procedure, and this is made especially manifest when the deduction is carried out on strings of symbols.

APPENDIX

Logic is the study of relationships between statements.
Statements in logic are also called **sentences**, or **propositions.** (Here, we'll call them sentences.)
A *sentence* is any assertion of fact, or even non-fact. The following are examples of sentences:

Snow is white.
London is the capital of England.
Snow is black.

One of the first and most important facts you'll have to understand about logic, is that logic *disregards what it is that a sentence asserts.* To a logician, a sentence such as "snow is white" is nothing more than a blank assertion, an X. The *content* of the sentence (that is, its meaning) is totally irrelevant to logic. For that reason, there is no loss if we simply denote sentences by single letters, such as A or B. Remember: Logic is the study of the *interrelations* between sentences, and it disregards what it is that specific sentences assert.

Logic is, in many ways, like arithmetic or algebra. Arithmetic deals with numbers; it studies the properties and interrelationships between numbers. The laws of arithmetic are equations such as $a + b = b + a$, which are true for all numbers a and b. Logic, in comparison with arithmetic, deals with *sentences*. Logic examines how compound sentences can be formed from simple ones, and how they interrelate. The laws of logic can be expressed as identities like

$$\text{A and B} \equiv \text{B and A}$$

(where ≡ means "is equivalent to"). Compare the above to the identity $a + b = b + a$ in arithmetic As we move on, you'll see many similarities between logic and arithmetic.

In arithmetic, you connect numbers with one another by operations such as +, − , × and ÷. Similarly, in logic you connect sentences with one another by using the following operations:

Operation	Symbol
A and B	$A \wedge B$
A or B	$A \vee B$
If A, then B	$A \rightarrow B$
Not-A	$\neg A$

Now, every sentence is either true or false. More importantly, the truth or falsity of any compound sentence is completely determined by the truth or falsity of each of its component sentences. For example, if we are told whether A is true or false, and also whether B is true or false, then we can determine whether $A \wedge B$ is true or false, whether $A \rightarrow B$ is true or false, and so on. It is convenient to express this information in the form of tables, called *truth-tables:*

(Truth-tables for the four connectives)

Compound formulas are called *equivalent* if they have the same truth-tables. For example, the formulas $A \rightarrow B$ and $\neg A \vee B$ are equivalent, and we symbolize this fact by writing

$$A \rightarrow B \equiv \neg A \vee B$$

A few such equivalences are of great importance in logic, and are listed next. (Equivalences).

Logical Deduction By examining the truth-tables, we can also see that certain logical *rules of deduction* are valid.A logical proof consists of the following: One or two formulas (or possibly more) are given to us initially, and we are told that they are true. From these initial formulas, called premisses, we logically derive conclusions. In each step of the proof, we use a rule of deduction to conclude that a statement is true. For example, we conclude that B is true, if we have already shown that A and $A \rightarrow B$ are true. The following are commonly used as rules of deduction: (Rules appear here).

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