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Structures: Categorical and Cognitive

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Abstract

This paper is an inquiry into two paradigms of structures: cognitive and categorical. This investigation comes from the two special interests of the author - mathematics and learning - not just learning mathematics. Insight into cognitive and categorical structures is what sometimes sets them apart from each other, but more often, it brings them closer together.

1 "Structure is what structure does." (Van Hiele, [11])

Jean Piaget [12], a Swiss biologist and psychologist, developed an influential model of child development and learning based on the idea that the developing child constructs increasingly sophisticated cognitive structures – moving from a few inborn reflexes such as crying and sucking to highly complex mental activities. Cognitive structure is a person's internal mental "map," a scheme or a network of concepts for understanding and responding to physical experiences within his or her environment. Schema is viewed as a connected collection of hierarchical relations, an organized structure of knowledge, into which new knowledge and experience might fit. Understanding of something is equated with assimilating it into an appropriate schema. The formation of schema is the brain organizing its own activity. Piaget's theories of the development of logico-mathematical structures are based on this reflective activity of the brain.

Gestalt theory emphasizes higher-order cognitive processes. The focus is the idea of 'grouping' – characteristics of stimuli cause us to structure or interpret a visual domain or problem in a certain way. The primary factors called the laws of organization, that determine grouping, are: proximity, similarity, closure, and simplicity. These factors can be explained in the context of perception and problem-solving. The essence of successful problem-solving behavior according to Wertheimer [14] is being able to see the overall structure of the problem. Directed by what is required by the structure of a situation for a crucial region, one is lead to a reasonable prediction, which like the other parts of the structure, calls for verification. Two directions are involved: getting a whole consistent picture, and seeing what the structure of the whole requires for the parts.

Piaget gave the following properties of structure: structure has totality, structure is achieved by transformations, and structure is auto-regulating.The van Hiele theory [13] puts forward a hierarchy of levels of thinking: visualization, analysis, informal deduction, deduction, and rigor. Van Hiele claims parallelism with Gestalt psychology by explaining that insight exists when a person acts adequately with intention in a new situation . He further describes that the most important property of structure is that structure can be extended because of its composition: "Structure is what structure does."

The above is a brief overview of a few structures and some of their properties, usually mentioned in discussions about learning and teaching.

2 Categorical Structures

A category $\mathcal{K}$ consists of a collection of *objects* and a collection of *arrows* (morphisms) satisfying certain conditions. Given arrows $g : X \to Y$ and $f : Y \to Z$ there is an arrow $f \circ g : X \to Z$ which we call the *composition* of f and g. For each object X there is an arrow $id_X : X \to X$, called the *identity on* X. The axioms for a category are:

$$\text{Composition is associative: } (f \circ g) \circ h = f \circ (g \circ h).$$
$$\text{Identity property: Given any } f : Y \to Z,\ \ f \circ id_X = f \text{ and } id_Y \circ f = f\ .$$

Category theory provides a consistent treatment of the notion of mathematical structure. Almost every known example of a mathematical structure with the appropriate structure preserving map yields a category. The classic example of category theory is Set, the category with sets as objects, functions as arrows, and the usual composition. What characterizes a category is its arrows and not its objects. Thus, the category of topological spaces with open maps is a different category than the category of topological spaces with continuous maps.

In all these cases the arrows are actually special sort of functions. That need not be the case in general: any entity satisfying the conditions given in the definition is a category. For example, an *ordered set* is a category with its elements as objects and one arrow for each $X \leq Y$, but none otherwise. A *deductive system* such that the entailment relation is reflexive and transitive is a category [10].

2.1 Universals

Category theory unifies mathematical structures in a second, and perhaps even more important, manner. Once a type of structure has been defined, it becomes essential to determine how new structures can be constructed out of the given one and how given structures can be decomposed into more elementary substructures. For instance, set theory allows us to construct Cartesian product. For an example of the second sort, given a finite abelian group, it can be decomposed into a product of some of its subgroups. In both cases, it is necessary to know how structures of a certain kind combine. The nature of these combinations might appear to be considerably different when looked at from too close. Category theory reveals that many of these constructions are in fact special cases of objects in a category with what is called a "universal property". From a categorical point of view, a Cartesian product, a direct product of groups, a product of topological spaces, and a conjunction of propositions in a deductive system are all instances of a categorical concept: the categorical product.

The universal property of product: Any arrow $h : W \to X \times Y$ from an object W is uniquely determined by its composites $p \circ h$ and $q \circ h$, where $p : X \times Y \to X$ and $q : X \times Y \to Y$ are 'projections'. Conversely, given W and two arrows $f : W \to X$ and $g : W \to Y$ there is a unique arrow h which makes the corresponding diagram commute, namely $h = (f, g)$.

Thus, given X and Y, (p, q) is "universal" because any other such pair (f, g) factors uniquely (via h) through the pair (p, q). This property describes the product $(X \times Y, p, q)$ uniquely (up to a bijection).

Many properties of mathematical constructions may be represented by universal properties of diagrams [10].

2.2 Functors: Trans-structuring

Another crucial aspect of category theory is that it allows to see how different kind of structures are related to one another. For instance, in algebraic topology, topological spaces are related to groups by various means (homology, cohomology, homotopy, K-theory). It was precisely in order to clarify how these connections are made that Eilenberg and MacLane invented category theory [10]. Indeed, topological spaces with continuous maps constitute a category and similarly groups with group homomorphisms. In the very spirit of category theory, what should matter here are the arrows between categories. These are given by functors and are informally structure preserving maps between categories.

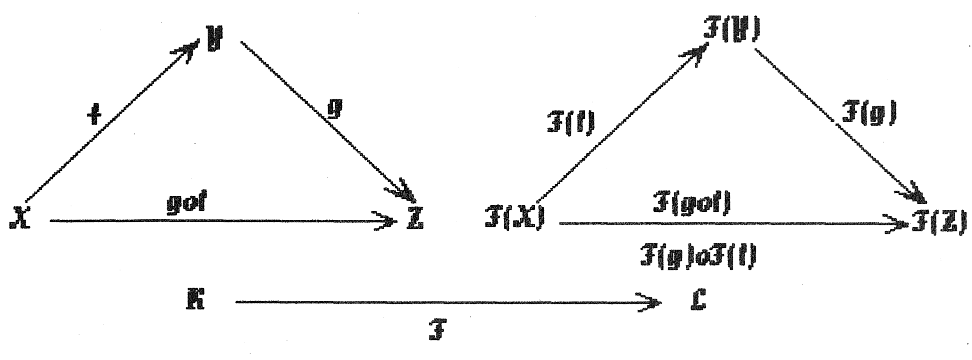

Figure 1: Functor $\mathcal{F}$: $\mathcal{K} \to \mathcal{L}$

A *functor* from $\mathcal{K}$ to $\mathcal{L}$ is a function from class of arrows of $\mathcal{K}$ to the class of arrows in $\mathcal{L}$ preserving identities and composition (see Figure 1):

$$\text{If } id \text{ is a } \mathcal{K}\text{-identity, then } F(id) \text{ is an } \mathcal{L}\text{-identity.}$$
$$F(f \circ g) = F(f) \circ F(g), \text{ where } g \in Hom(X, Y) \text{ and } f \in Hom(Y, Z)$$

It follows immediately that a functor preserves commutativity of diagrams between categories. Homology, cohomology, homotopy, K-theory are all example of functors.

2.3 One More Bridge: Natural Transformation

There are in general many functors between two given categories and it becomes natural to ask how they are connected. For instance, given a category $\mathcal{K}$, there is always the identity functor from $\mathcal{K}$ to $\mathcal{K}$ which sends every object of $\mathcal{K}$ to itself and every arrow of $\mathcal{K}$ to itself. In particular, there is the identity functor over the category of sets.

Suppose we have two functors F and G from the category $\mathcal{K}$ to the category $\mathcal{L}$ and an arrow $f : X \to Y$ in $\mathcal{K}$. A *natural transformation* n from F to G consists of a family of arrows, an arrow $n(X) : F(X) \to G(X)$ for each object X in $\mathcal{K}$, such that $n(Y) * F(f) = G(f) * n(X) : F(X) \to G(Y)$ (the corresponding diagram commutes for every such f).

An example would be "abelianization", which maps a group H to the abelian group $H/[H, H]$. If F were the fundamental group and G were the first homology group, we could say that abelianization is a natural transformation from F to G [10].

2.4 Is That All?

The above notions constitute the elementary concepts of category theory. However it should be noted that they are not fundamental notions of category theory. These are arguably the notions of limits/colimits which are, in turn, special cases of what is certainly the cornerstone of the theory, the concept of *adjoint functors.* We will not present the definition here. Adjoint functors permeate mathematics and this quality of spreading through is certainly one of the mystifying facts that category theory reveals about mathematics and probably thinking in general. Universality may also be described in terms of adjoint functors expressing the objective dialectical equilibrium.

In this manner, category theory provides means to circumscribe and study what is universal in mathematics and other scientific disciplines. Also, by identifying logic with the study of what is universal, category theory supplies the means to describe such logic, the objective logic of the discipline in question [10].

3 Cognitive Structures

Cognitive science is the interdisciplinary study of mind and intelligence, embracing philosophy, psychology, artificial intelligence, neuroscience, linguistics, and anthropology. Its intellectual origins are in the mid-1950s when researchers in several fields began to develop theories of mind based on complex representations and computational procedures. Its organizational origins are in the mid-1970s when the Cognitive Science Society was formed and the journal Cognitive Science began. Cognitive science is revolutionizing our understanding of ourselves by providing new accounts of human rationality and consciousness, perceptions, emotions, and desires. It explicates how structures of different kinds are related to one another as well as the universal components of a family of structures of a given kind. Philosophically, it can be thought of as constituting a theory of concepts. It also gives new prospective on some traditional philosophical questions, for instance on the nature of reference and truth.

Modeling the mind as an information-processing machine is a result of a large body of research about computer knowledge. It has led to much psychological and pedagogical insight, but has sharply limited ability to predict human behavior and learning nonetheless. Cognitive theory describes memory storage and recall structures resembling Piaget's schema and describes regular inclusions and revisions to these structures similar to Piagetian assimilation and accommodation [11].

Cognitive science raises many interesting methodological questions: What is the nature of representation? What role do computational models play in the development of cognitive theories? What is the relation among apparently competing accounts of mind involving symbolic processing, neural networks, and dynamical systems?

3.1 Mental Representations

There is much disagreement about the nature of the representations and computations that constitute thinking. Thinking can best be understood in terms of representational structures in the mind and computational procedures that operate on those structures. This central hypothesis of cognitive science is general enough to encompass the current range of thinking in cognitive science, including connectionist theories which model thinking using artificial neural networks [11]. Most work in cognitive science assumes that the mind has mental representations analogous to computer data structures, and computational procedures similar to computational algorithms. The mind contains such mental representations as logical propositions, rules, concepts, images, and analogies, and uses mental procedures such as deduction, search, matching, rotating, and retrieval. "Connectionists" have proposed novel ideas about representation and computation that use neurons and their connections as inspirations for data structures, and neuron firing and spreading activation as inspirations for algorithms. Cognitive science then works with a complex 3-way analogy of mind, brain, and computers. Mind, brain, or computation can each be used to suggest new ideas about the other two. Different kinds of computers and programming approaches suggest different ways in which the mind might work. Therefore, there is no single computational model of mind. The first computers are serial processors, performing one instruction at a time, but the brain and some newly developed computers are parallel processors, capable of doing many operations at once.

Philosophy, in particular philosophy of the mind, is part of cognitive science. From a naturalistic perspective, philosophy of the mind is closely tied in with theoretical and experimental work in cognitive science. Metaphysical conclusions about the nature of mind are to be reached, not by a priori speculation, but by informed reflection on scientific developments in fields such as computer science and neuroscience. Similarly, epistemology is not a standalone conceptual exercise, but depends on and benefits from scientific findings concerning mental structures and learning procedures. It is an empirical conjecture that human minds work by representation and computation. Although this computational-representational approach to cognitive science has been successful in explaining many aspects of human learning, critics of cognitive science have offered such challenges as: the emotion challenge, the consciousness challenge, the world challenge, the social challenge, the dynamical systems challenge, and the mathematics challenge [9].

3.2 Architecture

A very restricted family of structures that provide the frame within which the cognitive processing in the mind take place is called architecture [11]. If the consideration is restricted only to symbolic structures, architecture is closer to structures analyzed in computer science. Viewing the world as constituted of systems whose behavior is observed is part of the common conceptual apparatus of science - a system of given structure producing behavior that performs a given function in the encompassing system. The notion of architecture in this sense supplies the concept of the system that is required to attain flexible intelligent behavior by invoking most of the psychological functions - perception, encoding, retrieval, memory, composition and selection of symbolic responses, decision making, motor commands and actual motor responses.

The role of architecture in cognitive science is to be the central element in a theory of human cognition. A theory of the architecture is a proposal for the total cognitive mechanism. It is reasonable to say that the cognitive architecture is realized in neural technology and that it was created by evolution.

3.3 The nature of cognitive architecture: Functions

Since the architecture is defined in terms of what it does for cognition, the nature of the cognitive architecture is given in terms of functions, rather than structures and mechanisms. What are some functions defining this nature of cognitive science?

Memory - composed of structures that contain symbol tokens.

Symbols - patterns that provide access to distal symbol structures

Operations on symbols - processes that take symbol structure as an input and produce (compose) new symbol structures as output; a sequence of symbol operations occurs on specific symbol structures.

Operations - can construct symbol structures that can be interpreted to specify further operations to construct yet further symbol structures.

Interpretation - processes that take symbol structure as an input and produce behavior by executing operations

Interaction with the external world - perceptual and motor interfaces; real time demands for action

One additional consideration that is specific to the nature of human cognition is that *the human mind* can carry out a large number of constructions that seem very natural and so universal that they must be severely constrained [9].

4 Conceptual Tool: Categorical Logic

Natural logic involves the simultaneous addressing of constancy and change, because change is realizable only relative to constancy. In the logic of types and kinds, predicates and modal connectives express change. Categorical logic with inherent functoriality, is appropriate to deal with such constancy and change. In the same manner, categorical logic is the convenient mathematical environment for handling that functor between linguistic structures and the structure of the non-linguistic well as interpretation.

The categorical logic, the study of logic with the help of categorical means, has produced many important results. Suffice it to mention the generalization of Kripke-Beth semantics for intuitionistic logic to sheaf semantics by Joyal. Ellerman 1987 has tried to show that category theory constitutes a theory of universals which has properties radically different from set theory considered as a theory of universals. If we move from universals to concepts in general, we can see how category theory could be useful even in cognitive science. Indeed, Macnamara and Reyes have already tried to use categorical logic to provide a different logic of reference [9].

The logic of types and kinds takes on a mathematical life of its own, it retains a structural harmony of its own. It is discovered by studying the category of kinds.The logic of kinds is expressed in terms of categorical logic. Categorical logic is intuitionist, though it can also be classical when the occasion arises. Natural logic is many-sorted, and categorical logic, because it treats arrows as basic, seems specially adapted to deal with many-sorted systems.

4.1 Topos

The 'element-free' formulation of mathematics provided by category theory is most strikingly realized by applying it to set theory itself. Formally, an *elementary topos* $\mathcal{E}$ is a finitely complete category with exponentials and a subobject classifier Ω. A *subobject classifier* or truth-value object in a category with a terminal object is an object Ω together with an arrow$\top : 1 \to \Omega$ called the truth arrow, from the terminal object 1, such that the diagram has a universal (pullback) property: For each arrow $m : B \to A$ there is a unique arrow, called the characteristic arrow of m, $\chi(m) : A \to \Omega$ such that $\top \circ b = \chi(m) \circ m$.

More precisely, (1) $\mathcal{E}$ has pullbacks and a terminal object (and therefore, all finite limits), (2) $\mathcal{E}$ is cartesian closed, and (3) $\mathcal{E}$ has a subobject classifier.

In any topos $\mathcal{E}$ one can give natural definitions of arrows ('logical operations' in $\mathcal{E}$), $\sim: \Omega \to \Omega$; $\vee$, $\wedge$, $\Rightarrow$: $\Omega \times \Omega \to \Omega$ in such a way that, if we regard these arrows as algebraic operations on Ω, the resulting (Heyting) algebra satisfies the laws of intuitionistic propositional logic. In this sense intuitionistic logic is 'internalised' in a topos. With some justice, then, we may regard a topos as an instrument for reducing logic to mathematics, the remarkable thing being that the logic obtained is not (in general) classical, but intuitionistic. Thus category theory, far from being in opposition to set theory, ultimately enables the set concept to achieve a new universality [6].

4.2 Cognition $\leftrightarrows$ Categorical Logic

Category theory can also be viewed as a foundational discipline capable of clarifying and sometimes expanding our understanding of mathematical knowledge and its applications. This change is mainly to the discovery of F. W. Lawvere that some categories may be viewed as universes of variable/cohesive sets, capable of modelling theories that lack models in the more rigid universe of constant sets. Already, category theory has been applied to a variety of subjects ranging from physics to linguistics.

Category theory could be a conceptual tool in the study of cognition. The thesis is that the explicit adequate development of the science of knowing will require the use of the mathematical theory of categories [8].

Category theory has developed a variety of notions in order to provide a guide to the complex constructions of the concepts and their interactions which grow out of the study of space and quantity. Galileo's insight is that physics and mathematics mutually constrain each other; Chomsky's insight is that psycholinguistics and linguistics constrain each other. J. Macnamara suggests that cognition and logic constrain each other in the same manner and he is contributing that insight to the father of logic, Aristotle. This relation can be expressed as Cognition $\rightleftarrows$ Logic. And, in the context of this paper, without collapsing to a single subject, **Cognition $\leftrightarrows$ Categorical Logic.**

Macnamara [9] shares his vision that in cognition we are in the year 1690. Our calculus (categorical logic) has been invented. A deep and satisfying theory of the human mind will be developing and replacing tendencies in 'cognitive studies' that are unworthy of their subject.

References

[1] M. Alagic, *Abstraction, Reflection, and Learning* (abstract), Bridges - Mathematical Connections in Art, Music, and Science, Conference Proceedings, pp. 301. 1999.

[2] M. Alagic, *A Visual Presentation of Rank-Ordered Sets*, Bridges - Mathematical Connections in Art, Music, and Science, Conference Proceedings, pp. 237-244. 1998.

[3] S. Alagic & M. Alagic, *Order-sorted Model Theory for Temporal Executable Specifications,* Theoretical Computer Science, Vol. 179, no. 1-2, pp. 273-299. 1997.

[4] M. Alagic, & S. Neal, *From Concrete to Abstract: Reaching for Higher-Order Thinking Skills Through Geometry.* Paper presented at the NCTM Central Regional Conference, Topeka,1999.

[5] J. L. Bell, *Category theory and the foundations of mathematics*, British Journal of Philosophy of Science, Vol. 32, pp. 349-358. 1981.

[6] J. L. Bell, *Toposes and Local Set Theories, Clarendon Press,* Oxford, 1988.

[7] L. D. English, *Mathematical Reasoning Analogies, Metaphors, and Images*, Lawrence Erlbaum Inc. 1997.

[8] F. W. Lawvere, *Tools for the Advancement of Objective Logic: Closed Categories and Toposes,* In J. Macnamara & G.E.Reyes

[9] J. Macnamara & G. E. Reyes, *The Logical Foundations of Cognition*, Oxford University Press, New York, 1994.

[10] S. Mac Lane, *Categories for the Working Mathematician*, Springer Verlag, New York, 1971.

[11] A. Newell, P. S. Rosenbloom, & J. E. Laird, *Symbolic Architecture for Cognition*, In Foundations of Cognitive Science. M. I. Posner, (Ed.). MIT Press, Cambridge. 1989.

[12] J. Piaget & B. Inhelder, *The Child's Conception of Space*, Norton, New York, 1967.

[13] P. Van Hiele, *Structure and Insight A Theory of Mathematics Education.* Academic Press, Inc.1986

[14] M.Wertheimer, *Productive Thinking* (Enlarged Ed.). Harper & Row, New York 1959.