Bridges
Mathematical Connections
in Art, Music, and Science

Saccades and Perceptual Geometry:
Symmetry Detection through Entropy Minimization

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Introduction

The world surrounding us is composed of objects. Object recognition is an important factor in our survival and functionality in the world. Through our senses, we learn the properties of objects and distinguish them from one another. We learn, in particular, that objects occupy "volume" and are bounded by "surfaces", the type of entities whose existence and properties are learned through a combination of senses. Eventually, our visual perception of the external world relies on our ability to distinguish various pieces of surfaces, to integrate collections of surfaces into parts of an object, and to fill any missing information by inference and other mechanisms that develop as part of our survival strategy. Thus, a theory of visual perception of surfaces is at the heart of any comprehensive theory of human perceptual organization. First studied by Gestalt psychologists early in this century, perceptual organization concerns how retinal images are structured, how the various regions and elements are perceived as being related to each other in terms of part-whole relations, as well as various geometrical relations. Among the most pervasive and difficult problems in vision science are the nature of perceptual organization and the mechanisms responsible for it.

Perceptual geometry is an emerging field of interdisciplinary research whose objectives focus on study of geometry from the perspective of visual perception, and in turn, apply such geometric findings to the ecological study of vision. Perceptual geometry attempts to answer fundamental questions in perception of form and representation of space through synthesis of cognitive and biological theories of visual perception with geometric theories of the physical world. In our previous papers (Assadi 99), we have proposed a basic mathematical model for the Gestalt of surfaces, that is, the simplest phase of global perception of surfaces in the environment, as opposed to visual perception in laboratory and psychophysical experiments under controlled parameters. The result at this point is a computational model for perception of form (or Gestalt) of surfaces in natural environments, including those that have rough and non-smooth small-scale structure but with a perceived global (larger-scale) geometric form. Examples include grass and meadow, surfaces textured with sand-paper, natural scenes having rough texture such as the skin of crocodile, pine cones, a field of sea urchins, forests, etc.

The problems of figure-ground separation and scene segmentation in perceptual geometry could be formulated in terms of structural regularity of regions of images in statistical and information theoretic terms. Intuitively, as well as in psychophysical studies performed by cognitive scientists (e.g. Palmer, Rock, Tyler), perception of local structural regularity is fundamentally correlated with perception of local symmetry of surfaces, and under parallel projection of planar surfaces, with local symmetries of their images. In other words, such local symmetries distinguish prevalent regularity of common surfaces in the environment from randomness in arbitrary composition of colored dots, or what is the same, they distinguish between a meaningful image versus a generic pattern of a totally random selection of light intensities in matrices encoding local incoherence in optical properties. From a mathematical point of view, it can be shown that in the space of all possible patterns of light (i.e. all large matrices of same size with non-negative coefficients), the set of possible images of natural scenes is a very small subset. In the technical jargon of mathematical analysis, true images form a subset of measure zero in the space of all possible 2-dimensional patterns representing arbitrary light intensities.

We use the term symmetry in reference to all kinds of transformations that leave invariant some form of geometry, together with the related concepts such as harmony. Therefore, similarity in Euclidean plane geometry is a form of symmetry (in the so-called conformal geometry, where angles are preserved) although it is not necessarily a rigid motion. The term quasi-symmetry can be used for perceived regularity of structure that is compelling in its organization, but fails to be a strict symmetry in the mathematical sense above. This aspect of our research has a long-term history in psychology as the investigation of cognitive processes that underlie human perception of geometric forms, much the same way that Henri Poincare' posed in his 19-th Century treatise Science and Hypothesis and led to his construction of non-Euclidean geometry and optics [19].

Among many researchers studying symmetry, S. Palmer and I. Rock have contributed to our understanding of the role of Gestalt in perceptual organization in the context of symmetry, (e.g. Palmer, 1977; Palmer & Rock, 1994), and the formation of those aspects of our research that could belong to the domain of Gestalt Psychology, or mathematically speaking, passage of local-to-global properties in the context of geometric form. One of the main topics investigated by Palmer concerns the perception of internal geometrical structure in 2-D figures, including symmetry (Palmer, 1985; Palmer & Hemenway, 1978), part-whole relations form vs. texture (Kimchi & Palmer, 1982), figural goodness (Palmer, 1991), object-centered reference frames (Palmer, 1990), and grouping (Palmer, 1983; Palmer & Rock, 1994). In all these research areas, Palmer's approach has been to develop objective behavioral methods that can be used to study the stimulus factors that influence perceptual phenomena. In many cases, his theories have taken a transformational approach to problems of perceptual organization, where he analyzes phenomena in terms of the action of the similarity group (Palmer, 1982, 1983, 1991).

Biological Motivation

Our research has a long-term objective: to investigate the cognitive processes that underlie our perception of geometric forms. The time is ripe to address this question in the realm of cognitive neuroscience. While our proposed computational model serves to investigate and support the cognitive theories, it reveals the potential for a refined approach to cognitive and biological models of *visual information processing channels*. One could generally agree that the increasingly rapid pace of advances in our understanding of the biology of the brain and advances in computation power will open new ways for investigation of information processing in the brain. Thus, we are optimistic that in foreseeable future, we will have the scientific tools to understand neuronal substrates of low-level computations of visual, tactile, motor and auditory processes, which contribute to our perception of symmetry and regularity in structure.

This paper is a second computational attempt at modeling detection of repetitive patterns in visual perception, and potential reasoning about almost structural regularity in the presence of noise and natural imperfections of visual stimuli. Being as small a step as it may be, we propose a possible approach for visual perception of geometric form of surfaces endowed with symmetric patterns in their texture. The key biological observation is the dynamic nature of vision: visual perception of the physical world depends on saccades, the tiny, almost instantaneous jitters of the eyeballs (e.g. see pp. 78-80). Without such seemingly random motions of the eyes, the photoreceptor cells of the retina get chemically saturated from the steady invasion of photons, and the image on the retina ceases to exist! All neurons in the visual cortex have receptive fields (see Hubel 1995, page 41-43), defined, roughly speaking, as the cone-shaped region of the space measured in terms of the visual angle. This potentially stimulates a neuron through its variation of light intensity. The concept of the receptive field and its biological properties leads us to hypothesize that: there is a biologically realistic hybrid computational model of intermediate-level vision in which the process of neuronal detection of visual symmetry in the presence of repetitive patterns involves an adaptive series of comparison of patterns of spike trains. This is due to visual stimuli that are brought about by a sequence of saccades.

We refer to this statement as ***the Adaptive Saccades Hypothesis*** for detection of symmetry. The long-term goal of the present research includes formulation and verification of models that incorporate hypotheses such as the Adaptive Saccades Hypothesis above. Therefore, our first attempt focuses on modeling adaptive processes that simulate saccades and comparison of the resulting neural activation, in parlance of neural networks and learning theory (Latimer, 1996). In this paper, we consider a simplified version of the above-mentioned theory: a computational model is presented that detects the fundamental domain for translation symmetry of patterns on a flat surface parallel to the observer's plane of view (or the robot's camera screen). This model is robust with respect to noise, partial occlusion and some irregularity in the translation symmetry. Moreover, it lends itself to generalization to visual attention and theories of active vision. Finally, there is a generalization to hybrid adaptive implementations that incorporate support vector machines, a regression technique that has its biological justification in the finer anatomy of the superior colliculus. In a forthcoming paper, we present the generalization of the model below to incorporate support vector regression.

Perception of symmetry in the human brain has been the subject of various studies. The approach in neurobiology, the so-called bottom-up approach (see for example, Hubel 1995), proceeds to study the cascade of neuronal events from the onset of the stimulus as it progresses toward the emergence of the percept of symmetry (Julesz 1979; Troscianko 1987). In contrast, many experimental and cognitive psychologists take a "top-down" viewpoint (Rock 1963; Palmer 1978; Royer 1981; Van Gool 1990; Wagemans 1993). In the "top-down" construct one proposes a theory (based on the psychophysical measurement) that reconstructs the sequence of events in the brain leading to the observed behavior; see e.g. (Hogben 1976; Zimmer 1984; Herbert 1994). Alternatively one may propose a theory where both top-down and bottom-up reasoning, possibly with several building blocks of each kind. Such hybrid theories use each type of reasoning in a sequence that complements and improves partial results in individual steps. More generally, it is reasonable to propose a more integral (holistic) approach to study symmetry, for instance, by considering the higher-level cognitive functions, such as thinking, creativity, and problem solving (Van Gool 1990; Rentscher 1996; Varshney and Burrus 1997).

As a starting point, it is convenient, and not so unreasonable to begin with the assumption that any computational theory of perception of symmetry would most likely use the special cases of translation and reflection of planar periodic tilings as one of its building blocks; see e.g. (Royer 1981). As for reflection symmetries, there is a great deal of research that is partially outlined in (Tyler 1996) and elsewhere (e.g.(Gerbino 1991) (Wagemans 1992; Wagemans 1993; Wagemans 1996)). In fact, almost all articles in that volume are devoted to bilateral and reflection symmetry (e.g. (Marola 1989; Dakin 1996)) with a few exceptions dealing with circular and rotation symmetries (Palmer 1978). Translation symmetry seems, for the most part, not adequately covered in the literature (see (Corballis 1974; Bruce 1975; Baylis 1994)). As for connectionist models, back propagation model for detection of reflection symmetry are provided in (Baylis 1994; Latimer 1996). Dakin and Watt, on the other hand, have used spatial filters for detection of bilateral symmetry (Dakin 1996). Thus, for the remaining part of this paper, we concentrate on translation symmetry.

Review of Related Previous Results

In our previous work (Manske, et. al, 1999), detection of symmetry in periodic tiling was studied from the point of view of machine vision. The very systematic approach of the main algorithm to search and target selection in the image makes the model of (Manske, et. al, 1999) incompatible with the human visual search and natural eye movement. Nonetheless, a case was made for a dynamic model of symmetry detection where adaptive saccades in conjunction with visual attention play an important role. In this work, we present a computational model of saccadic target selection and simulate its action in the context of perception of global symmetry of tiling. This approach uses local (foveal) symmetry estimation via direct saccadic eye movements, in agreement with well-known biological models of human eye movement. However, the variant of spatial channels considered in this paper are motivated by wavelets and filtering, rather than the model based on spatial Fourier transform. From the point of view of computational efficiency and numerical convergence, the model is certainly superior to the Fourier alternative. While the computationally proposed approach has not been experimentally tested by biologists in the laboratory, the localized nature of wavelet supports appears to fit better with the Gabor-like filter models of receptive fields in the neural circuitry of low-level primate vision.

The motivation for a model of saccadic target selection finds its roots in the properties of the superior colliculus (SC). Studies of SC indicate that the nature of the information processing is highly distributed and integrated over a large spatial area. More specifically, the following is known: (1) In the SC, visual cells have large and overlapping receptive fields; (2) The SC has a layered architecture where retinotopic mapping is preserved; (3) The size of the visual fields increases systematically with depth of layers. It has been demonstrated that any particular point in the collicular motor map can be stimulated by visual input from a wide region of the visual space. Conversely, stimulation from a punctate peripheral source can generate activity over a wide area of the colliculus.

A Heuristic Discussion Of Symmetry Detection

We start with the consideration of the tiling of a rectangular subset of the plane **s** with the rectangular tile B. For a perfect (no partial tiles) and noiseless tiling, the regularity of the structure of this subset can be determined by solving the following problem: *Determine B with the smallest area such that* **s** *can be reconstructed from a number of horizontal copies and a number of vertical copies of* B. In this deterministic setting, the problem can be viewed as finding a basic element for **s** in the following way. The subset **s** is obtained form translates of this basic element B and the apparent large data volume of **s** can be compressed by finding the basic element B and constructing **s** from the translates of B. To address the same problem in the presence of noise and other imperfections, we must solve the search for this basic element in the statistical setting. In this setting we view **s** as a signal that we wish to represent in terms of B yet to be determined.

The traditional Fourier representation of a signal is based on the superposition of sinusoids. The sinusoids can be thought of as elements of a dictionary – a collection of parameterized waveforms. However, one may choose from a host of alternate dictionaries such as wavelets, chirplets, cosine packets and others. Each such dictionary ***D*** is a collection of elements ϕ_t, where *t* is a parameter, and is used to decompose a signal **s** in the form of:

$$s = \sum_t a_t \phi_t + C$$

Where C is a "error" term representing how well the signal s is approximated using the given dictionary. The goal of using alternative dictionaries is to obtain a sparse as well as robust representation of the signals. However, the sparseness requirement makes for a difficult optimization problem since minimizing the number of coefficients in the representation of s above requires a combinatorial check of all coefficient possibilities. Mallat and Zhang [15] proposed a general approach (called matching pursuit), based on the simple idea that one can start with an initial approximation and at each stage find the element in the dictionary such that its multiple minimized the residual C – a simple greedy algorithm. The algorithm is stopped after a few steps and the representation of the signal contains only a few elements of the dictionary. Of course, as it has been pointed out by Devore [6], this algorithm can fail badly for certain signals and non-orthogonal dictionaries. Basis pursuit as proposed by Chen [23] solves some of the problems by attempting to minimize the L^1 norm. However, most of the work mentioned so far and the reference therein considers the problem of choosing coefficients for a given dictionary.

The case of interest for us is to construct a (small) dictionary such that one of the items in the dictionary can be used to represent the information content of the entire signal. Loosely speaking, we wish to find an element for the dictionary such that any random selection of a segment of the entire signal effectively provides no additional information beyond what is available from the selected element from the dictionary. For this approach, we only borrow the spirit of the Mallat and Zhang algorithm in the sense that we start with an approximate solution and look for improving residual errors through iterations. However, the analogy stops at this level since our algorithm uses a multiscale search criteria and an information theoretic measure for its approach as will be described in the next section.

Principal component analysis (PCA) is a well-known statistical method for lowering the dimensionality of a data set by finding a set of basis, which contain "most" of the information in the original data set. To use the PCA approach to discover the tile B the single image object represented by s must be transformed into a collection of data. A straightforward approach would proceed as follows: *Choose a trial size for B and break s into a collection of trial Bs that will act as the collection of data for the PCA. Compare the results of PCA with those of another trial B and continue until a B is found that has the optimal size(this is reminiscent of Mallat's algorithm).* The computational difficulty with this approach stems from the large search space associated with the solution of the problem. There is also a conceptual difficulty which stems from our desire to use a dynamic process in which an approximate symmetry is detected and then continually refined - i.e. the use of coarse grain information to determine approximate symmetry and then refining by using fine-grained information.

We introduce a dynamic initial process based on entropy and operating on a "low-resolution" version of the signal, which can operate in parallel with "high-resolution" information to discover symmetries. Entropy is a measure of order and is related to PCA as we will see below. The initial process operates using the entropy measure for a continually refined coarse grained information. This process is used to determine the initial approximation for the size of B which is continually refined with the first process.

PCA and Entropy

Suppose that we have broken the rectangular domain D into a set of tiles. Each tile can be treated as a zero mean vector indexed by the variable t, X(t), define the covariance matrix R: $R = E\left[X(t)X^T(t)\right]$ (E denotes the expectation). Let $R = U\Lambda U^T$ be the eigendecomposition of R where $U = [u_1, \cdots, u_n]$ is an orthogonal matrix formed by eigenvectors $u_1, \cdots, u_n$, and $\Lambda = diag(\lambda_1, \cdots, \lambda_n)$ a diagonal matrix formed by eigenvalues of R. Then, in this new coordinate system, X(t) can be written as:

$$X(t) = a_1(t)u_1(t) + \cdots + a_n(t)u_n(t)$$

where $a_i(t) = X(t)u_i$, and the following properties are satisfied:

$$E[a_i(t)] = 0, \quad E[a_i^2(t)] = \lambda_i, \quad E[a_i(t)a_j(t)] = 0 \; for \, i \neq j$$

In this setting, given X(t), PCA provides a new coordinate system in which a_i are decorrelated. For the precise tile size any two adjacent blocks are maximally correlated. Therefore, decorrelation of the precise tile size means that a first eigen-value is significantly larger compared to the remaining eigen-values.

However, as we pointed out in the informal discussion, searching the large space of all tile sizes is not an efficient approach. To remedy the inefficiency we introduce a dynamic multi-scale process that uses entropy at a coarse grain level to obtain an initial approximation of the tile size. Given a set of discrete random variables (or symbols) denoted by x_i with probability of the event x_i denoted by $P(x_i)$, entropy is defined as:

$$H(x) = E[\log(P(x))] = -\sum P(x_i)\log(P(x_i))$$

The concept of mutual information between random variables x and y is related to the concept of joint entropy and is defined as:

$$I(x,y) = H(x) + H(y) - H(x,y) \quad where \quad H(x,y) = -E[E[\log(P(x,y))]]$$

where P(x , y) is the joint probability of events x and y. Notice that H(x , y) is zero when events x and y are statistically independent. It is important at this point to notice that when the distribution of the random variables follows a Guassian distribution, finding decorrelated components as done by PCA is the same as minimizing mutual information. Therefore, the two approaches of PCA and entropy are closely related. We also note that both methods are applicable to symmetric tiling without the presence of any projective transformations.

To use entropy in a dynamic process we require another ingredient. One must observe two important facts: 1) entropy as defined is invariant under a permutation of symbols. 2) entropy is subject to fluctuations when noise is present, 3) accurate computation of entropy requires the knowledge of distribution of the random variables. Although, the distribution is often taken to be synonymous with frequency, the use of frequency may introduce unacceptable inaccuracies. Furthermore, the invariance to permutations has the potential of giving false positives. We can overcome these problems in practice by introducing an filtering operator F. In the discrete case of images we are considering, the filtering operator replaces every pixel with an average of the pixels in a neighborhood:

$$F(i,j) = \frac{1}{(2s+1)(2t+1)} \sum_{l=-s,m=-t}^{l=s,m=t} F(i+l, j+m)$$

where s and t define the dimensions of the window over which averaging is performed. Since F captures information from a neighborhood it has the affect of reducing noise, reducing the affect of inaccuracies of estimating the distribution form the frequencies and reducing the affect of false positives resulting from permuted sets. Furthermore, from a biological view point, it is the statement that the coarsest level of information, that is what is available first, is used for estimates as soon as it is available.

Computational Formulation of Symmetry Detection

For our computational work we have used a series of images which contain both natural as well as synthetic symmetries. The images are stored as grayscale images with 256 levels of gray and are normalized to have intensity between values of zero to one. The algorithm is written in the Matlab computational environment and the core of the algorithm excluding the filtering portion runs very fast.

We describe how the overall process can be visualized in the following diagram where we have commented alongside the diagram.

Starting at the coarsest level of resolution we proceed to compute entropy for a window which is ¼ the size of the entire window.

First filter the image to the initial window size compute entropy for a random window location and repeat for a second random window.
if entropy continues to decrease, we want to continue with increasing resolution and measuring entropy,

If entropy begins to increase for both windows then we may be transitioning to a window of size smaller than the symmetric tile.

First we optimize the PCA process for the horizontal line scans and then optimize PCA for the vertical line scans.

```
Entropy_decreasing = true;
Initial_Window = 0.25*max(size(image));
P_Entropy = large_number;
while (entropy_decreasing)
        I = Filter with F in Initial_Window
        e1= entropy_of_random_window(I)
        e2= entropy_of_random_window(I)
        if e1 < p_entropy or e2 < p_entropy
                p_entropy = min(e1,e2);
                half the size of Initial_Window
        else
                if e1<e2 report e1
                else report e2
        endif

w = width(reported window);
w = optimize_PCA(w,I,'horizontal');
h = height(reported_window);
h = optimize_PCA(h,I,'vertical',w)
```

The simulation results are illustrated in the figures below. From our simulation we have found that the entropy portion of the algorithm always overestimates the size of the symmetry window. This allows the PCA portion of the algorithm to start with the window as an upper bound. We also found that the entropy portion of the algorithm works just as well with non-rectangular tiles. We use this fact later to discuss how non-rectangular tiles detected. Finally we mention that we determine the size of the tile and not necessarily the tile itself. The choice of the tile has cognitive components that need to be encoded in a higher level algorithm.

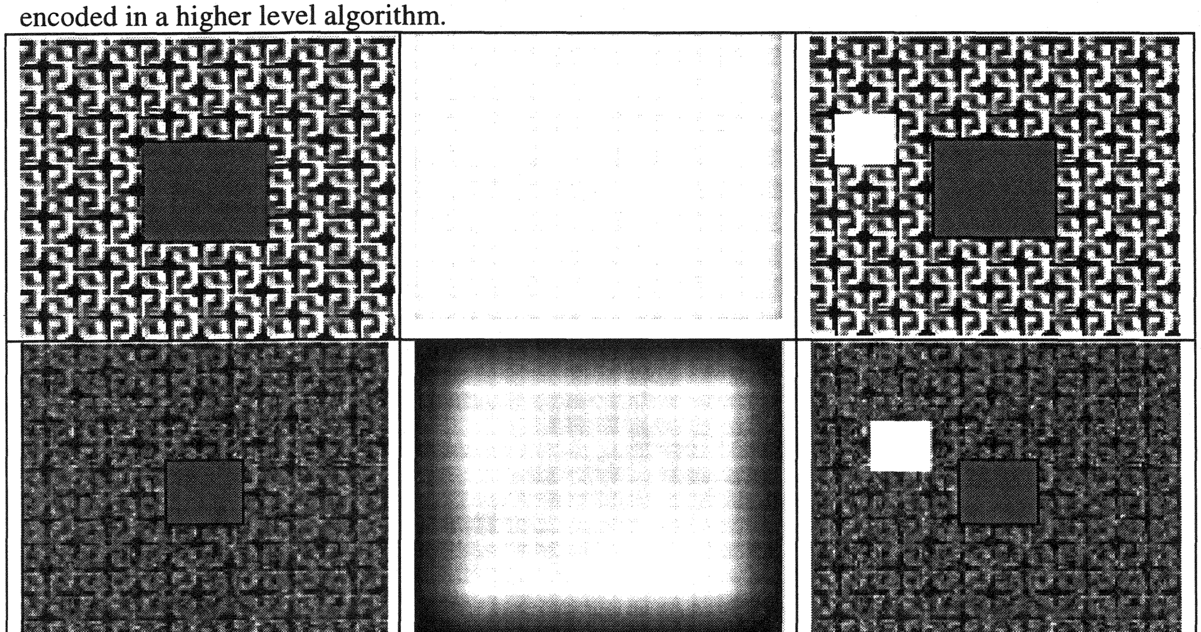

Figure 1.

Initial entropy minimization leads to an estimate of tile size. The tile is shown on the background of the image. Top image is the original. Bottom image is the noisy version of the top.	The entropy is measured after applying the averaging operator with decreasing window size. The two images show how images appear at the scale at which the tile size was determined.	Final estimate of the window size at the corresponding level of application for the averaging operator is shown in the smaller tile with the white background.

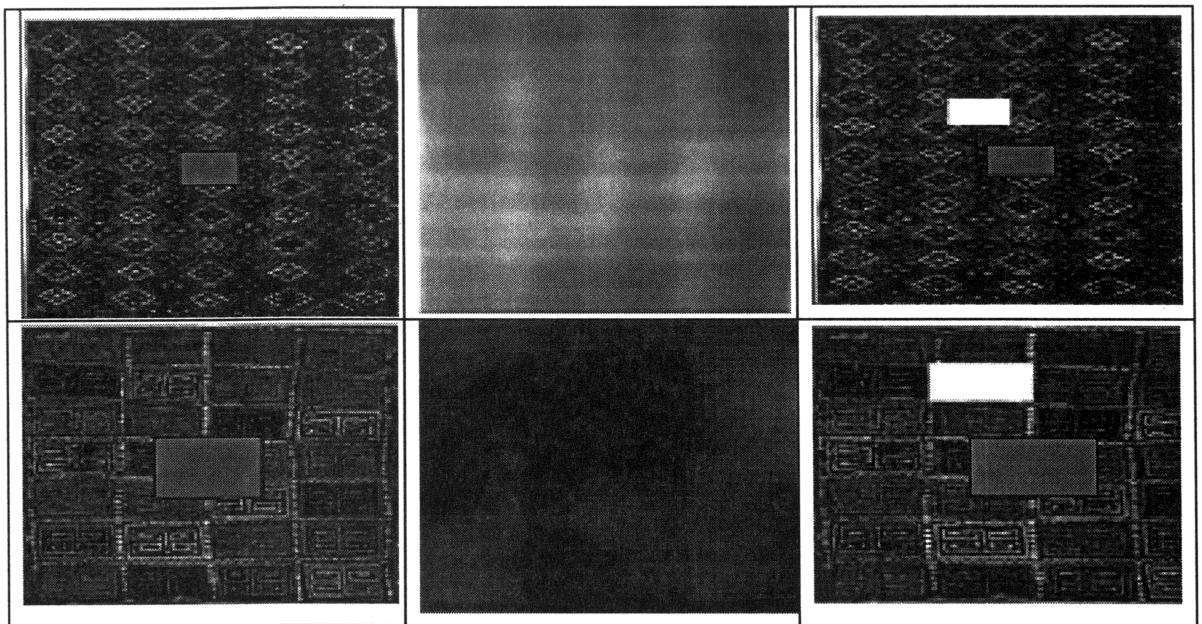

Figure 2.

Two images from handmade carpets were used for symmetry detection. Notice that these images are not perfectly tiled images.	The entropy measure is used on the image that appears as shown for corresponding carpets.	PCA to determine the final tile size is shown. Notice that in the top case the tile size is not exactly correct.

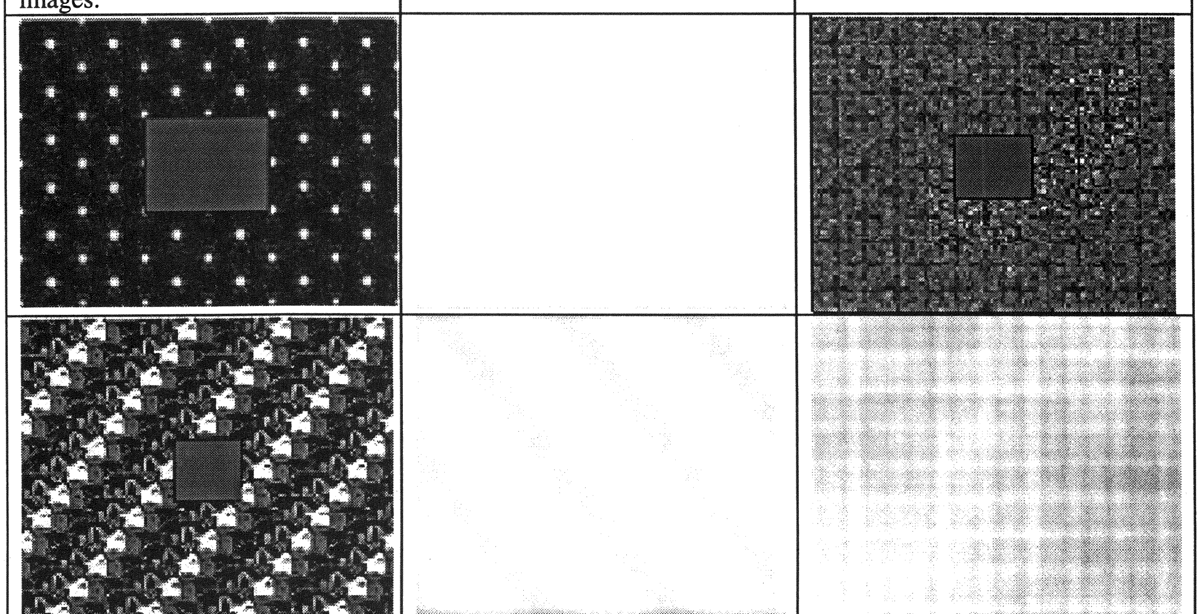

Figure 3.

Although as symmetric figures they perceptually appear not to have a clear rectangular region that generated it the entropy estimation shows a tile of reasonable approximation.	The images show what the entropy measure "sees" for computing entropies. Notice the clear presence of patterns.	These two images show one of the images in figure 1 that now is corrupted by non-uniform noise. Top shows the entropy estimate measured at the scale shown by the bottom image.

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