BRIDGES
Mathematical Connections
in Art, Music, and Science

Sections Beyond Golden

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Over the centuries, the Fine Arts have celebrated several special numerical ratios and proportions for their visual dynamics or balance and sometimes for their musical potential. From the golden ratio to the sacred cut, numerical relationships are at the heart of some of the greatest works of Nature and of Western design. This article introduces to interdisciplinary designers three newly-discovered numerical relations and summarizes current knowledge about them. Together with the golden ratio they form a golden family, and though not well understood yet, they are very compelling, both as mathematical diversions and as design elements. They also point toward an interesting geometry problem that is not yet solved. (Some of these ideas also appear in **[5]** in more technical language.)

I. Sections and proportions

First I want to clarify what is meant by golden ratio, golden mean, golden section and golden proportion. When a line is cut (sected) by a point, three segments are induced: the two parts and the whole. Where would you cut a segment so that the three lengths fit a proportion? Since a proportion has four entries, one of the three lengths must be repeated, and a little experimentation will show that you must repeat the larger part created by the cut. Figure 1 shows the golden *section* and golden *proportion*. The larger part, a, is the *mean* of the extremes, b and $a + b$.

As the Greeks put it: *The whole is to the larger part as the larger is to the smaller part.* To the Greeks this was a harmonious balance, an ideal asymmetry. Later, when geometry was being translated and relearned in Europe, medieval philosophers would divide a perception into perceiver and perceived and divide an ideal society into smaller ruling and larger serving classes. Still later, Renaissance artists would be drawn into many questions of means between extremes.

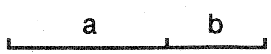
Ignoring a reflection of Figure 1 (b could be on the left and a on the right) this cut is unique, which can be proved if we find the golden *ratio* — the only possible value of $a{:}b$. The starred pentagons in Figure 2 illustrate the same proportion using similar triangles, naming the golden ratio ϕ (phi, pronounced *fee*). To derive a value for ϕ, follow the algebra suggested by Figure 2:

$$\frac{a}{b}=\frac{a+b}{a}=\frac{\phi}{1} \;\Rightarrow\; \frac{a}{b}=\frac{a}{a}+\frac{b}{a} \;\Rightarrow\; \phi=1+\frac{1}{\phi} \;\Rightarrow\; \phi^2-\phi-1=0.$$

The positive solution of this quadratic equation is $\phi = (1 + \sqrt{5})/2 \approx 1.618$, the golden ratio. Thus the section determines a single ratio.

The Greek fascination with proportion inspired a design tradition in which a harmonious arrangement of elements is defined as one that realizes some special ratio and repeats it in proportion. Often this repetition is potentially endless, as seen in the progressions of rectangles and triangles in Figure 3. There is a long tradition of religious art associated with the square root of 3 — a diagonal of the unit-sided hexagon — and Robert Lawlor's *Sacred Geometry* **[3]** describes some uses of this number. Renaissance architects identified properties of a diagonal of the unit-sided octagon, $\theta = 1 + \sqrt{2}$, later called the Sacred Cut and recently described by Kim Williams **[6]** (see Figure 4). These numbers and others are involved not only in human artifice but in the growth of natural forms.

What's new in ratio and proportion? The new discoveries are partial answers to the question: What proportions can be made by multiple cuts of a segment? If we cut twice, we create six segments: the three parts, the sums of the middle and the left or right part, and the whole. Is there some extended proportion that can be satisfied by six such lengths? The answer is in Figure 5. Quite by accident I found that the unit heptagon's diagonals form a 3-by-3 proportion that describes a *tri*section of a segment. It remained, then, to prove that this is best possible, that this is the unique optimally proportional trisection analogous to the golden bisection. By *optimal* I mean: just as there is no cut other than the golden section that fits a non-trivial proportion, so there is no pair of cuts other than the



$$\frac{a+b}{a} = \frac{a}{b}$$

Figure 1.
Golden section & proportion

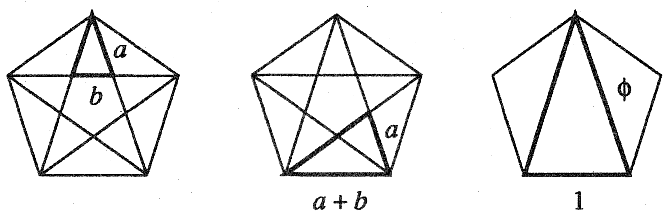


Figure 2.
Derivation of ϕ

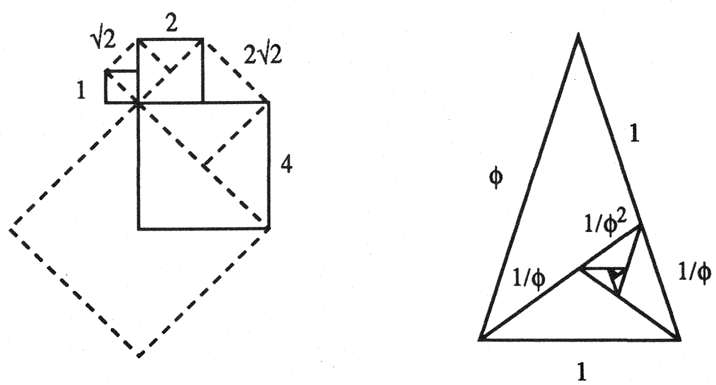
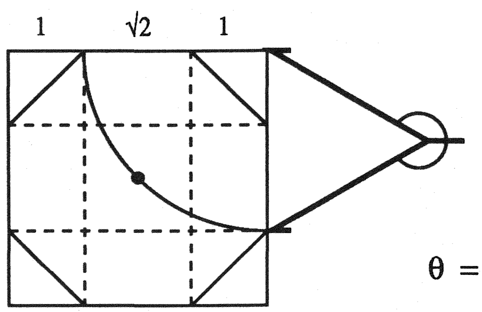


Figure 3.
Repeating proportions



$$\theta = \frac{1+\sqrt{2}}{1} = \frac{1+\sqrt{2}+1}{\sqrt{2}}$$

Figure 4.
Sacred Cut construction
of octagon

heptagonal type that yields an equal or greater harvest of proportions.

To explain this it will help, first, to explain what a 3-by-3 proportion is. A triple ratio — say 6:9:12 — might be a ratio of height-to-width-to-length of a shoebox. And one can see that a proportion of three triple ratios —

$$\begin{aligned} & 2:3:4 \\ = \; & 6:9:12 \\ = \; & 10:15:20 \end{aligned}$$

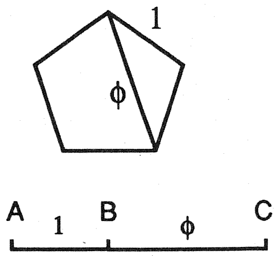
— suggests three geometrically similar boxes. To cross multiply such a thing (say, if you wanted to solve for an unknown), set a copy of the proportion on its right.

2	3	4	**2**	**3**	**4**
6	9	12	**6**	**9**	**12**
10	15	20	**10**	**15**	**20**

Find the six diagonals and multiply the three numbers along each diagonal.

$$2 \cdot 9 \cdot 20 = 3 \cdot 12 \cdot 10 = 4 \cdot 6 \cdot 15 = 4 \cdot 9 \cdot 10 = 2 \cdot 12 \cdot 15 = 3 \cdot 6 \cdot 20 = 360.$$

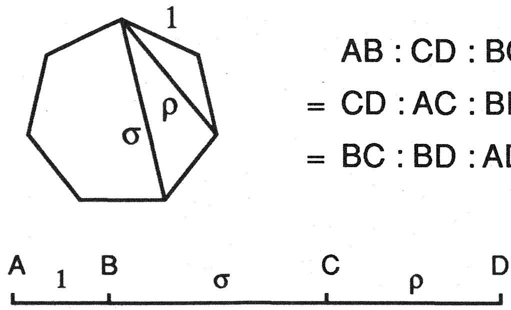
Figure 5. Optimal sections and proportions



AB : BC
= BC : AC

or

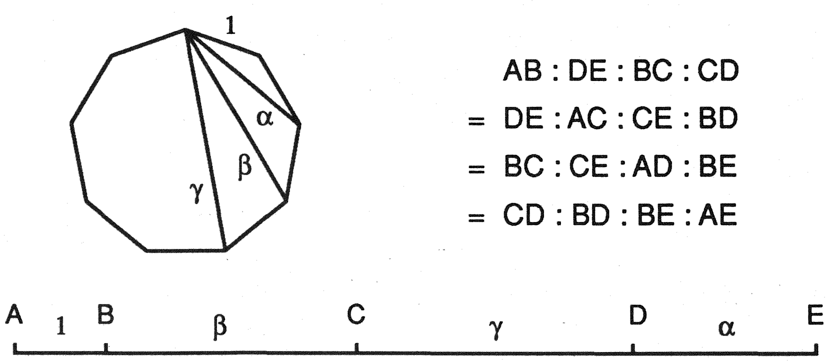
smaller : larger
= larger : whole



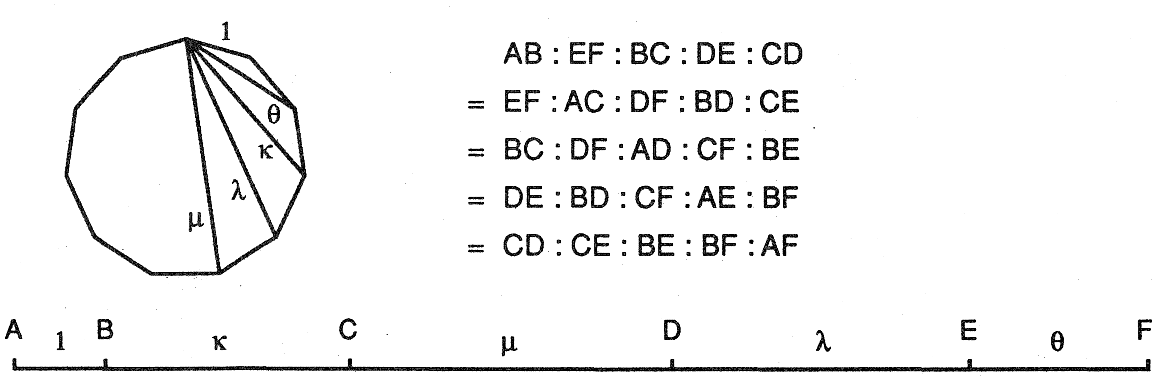
AB : CD : BC
= CD : AC : BD
= BC : BD : AD

or

small : medium : large
= medium : small + large : medium + large
= large : medium + large : whole



AB : DE : BC : CD
= DE : AC : CE : BD
= BC : CE : AD : BE
= CD : BD : BE : AE



AB : EF : BC : DE : CD
= EF : AC : DF : BD : CE
= BC : DF : AD : CF : BE
= DE : BD : CF : AE : BF
= CD : CE : BE : BF : AF

All six products are equal. (As we used to say in industrial design, there's nothing like a consistent product.)

To deal with the problem of uniqueness, it will help to simplify the notation a little. Let the parts of the golden section be 1 and x, with $1 < x$. Then

$$\begin{aligned} &1 : x \\ = \; &x : x+1. \end{aligned}$$

Cross multiplication reveals that $x^2 = x + 1$, or $x^2 - x - 1 = 0$, and we have already seen that $x = \phi$. Let the parts of a trisection be $1 < x < y$. These can be arranged in any of three orders, depending on which is in the middle of the segment, so there are three cases:

order 1, x, y	**order x, 1, y**	**order 1, y, x**
$1 : x \quad : y$	$1 : x \quad : y$	$1 : x \quad : y$
$= x : 1 + x : x + y$	$= x : 1 + x : 1 + y$	$= x : 1 + y : x + y$
$= y : x + y : 1 + x + y$	$= y : 1 + y : 1 + x + y$	$= y : x + y : 1 + x + y$

But cross multiplication and substitution reveal that the first two cases have no solutions, and the third (realized in Figure 5) uniquely determines that $x = \rho$ and $y = \sigma$.

Rho and sigma are solutions of cubic equations, and are not expressible exactly without using the square root of a negative. The convenient expression $(1 + \sqrt{5})/2$ for ϕ has no analog for ρ and σ. To find ρ and σ on the calculator, use trigonometry: as $\phi = 2\cos(\pi/5)$, so $\rho = 2\cos(\pi/7) \approx 1.80194$, and $\sigma = \rho^2 - 1 \approx 2.24698$.

Since ρ and σ are cubic numbers, the heptagon is not classically constructible (with compass and straightedge), which may explain the ancients' silence on the matter. The Greek geometers' method of investigation was construction. This and their limited understanding of irrational numbers would inhibit their analysis of figures like the heptagon. Archimedes at least constructed the heptagon with a *marked* straightedge and may have discovered more. (Dijksterhuis [1] cites evidence of a lost Archimedean manuscript entitled *On the Heptagon in a Circle*.) The derivation of ϕ and its properties by similar triangles (Figure 2) has been known since ancient times, and one would think that the Greeks would have applied the same reasoning to other figures despite their inconstructibility.

Figure 5 also shows that this trend continues, that the enneagon's diagonals — $1 < \alpha < \beta < \gamma$ — are involved in a 4-by-4 proportion describing the ten subsegments of a unique quadrisection. This time there are 16 cases to check, and one is realizable. The pentasection offers 125 cases, one of which is a 5-by-5 proportion using the diagonals of the unit 11-gon. This is the largest known unique optimally proportional section, and there is as yet no mathematical proof that uniqueness continues indefinitely. Ideas are welcome.

II. To Add is to Multiply

The numbers in this optimal family have remarkable properties. Most significant to artists is the property that allows similar figures to be arranged easily or allows a figure to be dissected into a set of similar figures. This is the origin of the Greek idea of geometric progression — that *multiplication can be accomplished instead by addition.* Similar figures are similar by virtue of a proportion, an equation based on multiplying and dividing, and if these can be accomplished by adding and subtracting, then the figures allow repeated similarity. For instance, it is well known that the square of the golden ratio is one more than itself (written above as $\phi^2 = \phi + 1$) and that one less than ϕ is its reciprocal ($1/\phi = \phi - 1$). The first relation equates multiplication with addition, while the second accomplishes division through subtraction. These relations generate one of the drawings in Figure 3, as well as the famous golden spiral or nautilus construction (shown in Figure 6 as a double spiral, the fiddlehead). The heptagonal ratios also behave in surprising ways — adding to multiply, subtracting to divide.

(A)			
	$\rho^2 = 1 + \sigma$	$\sigma/\rho = \sigma - 1$	$1/\sigma = \sigma - \rho$
	$\rho\sigma = \rho + \sigma$	$\rho/\sigma = \rho - 1$	$1/\rho + 1/\sigma = 1$
	$\sigma^2 = 1 + \rho + \sigma$	$1/\rho = 1 + \rho - \sigma$	

Another application of "to add is to multiply" is shown in Figure 7, which depicts a visual analog of the optimal 2-by-2 and 3-by-3 proportions, where the areas of the small rectangles are the proportions' entries. P. H. Scholfield, in his *Theory of Proportion in Architecture* [4], used the square with golden sected sides (Figure 7, top) to illustrate a problem in design economy. If a rectangle is dissected by one vertical and one horizontal line arbitrarily, then in general nine differently shaped rectangles (no two similar) are formed. There are many ways to reduce this number

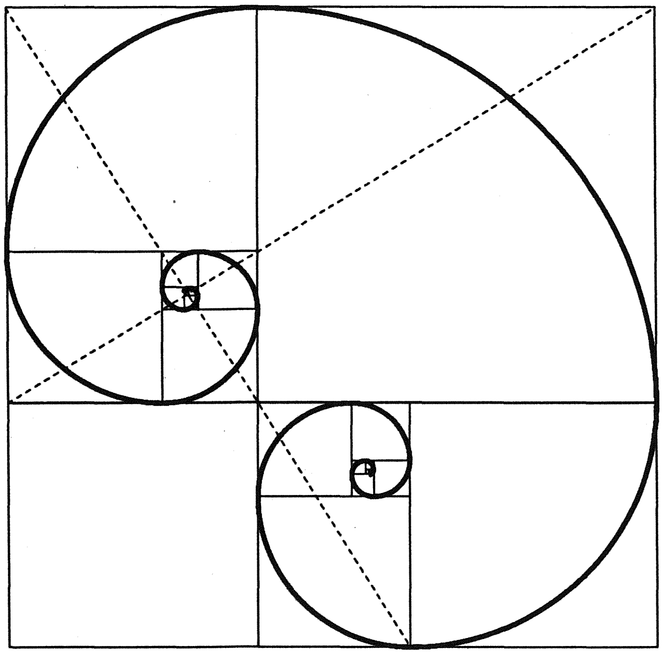


Figure 6. Golden double spiral (fiddlehead)

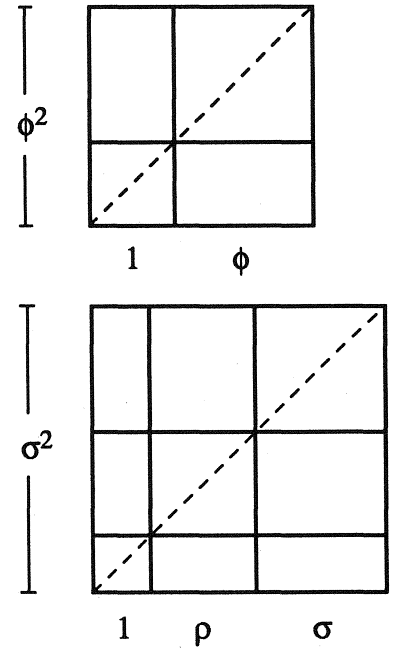


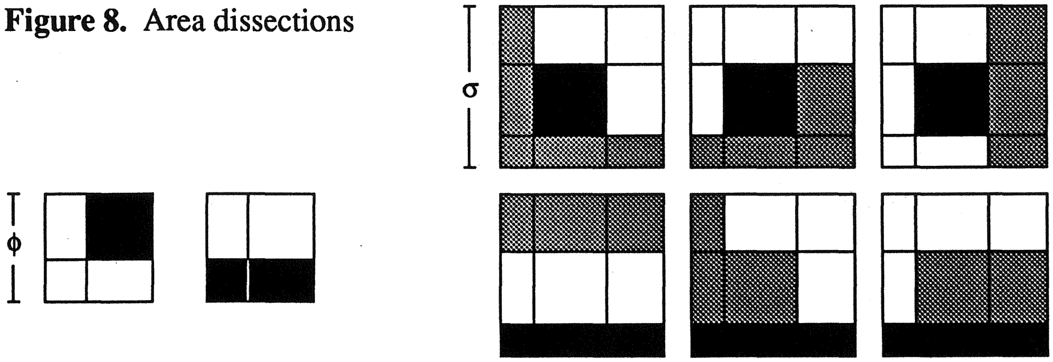
Figure 7. ϕ and σ dissections

by introducing symmetries in the figure. But if we are allowed only one symmetry, a diagonal reflection, then there is only one solution that yields the minimum of three different rectangles (three similarity classes). The ϕ^2-square in Figure 7 is the solution, and its rectangle types are 1-by-ϕ, 1-by-ϕ^2, and square.

Now if two vertical and two horizontal lines cut a rectangle, then in general 36 rectangles are formed. Scholfield points out that the sacred-cut square (Figure 4) reduces this to five non-similar rectangles, and the square with side length $\phi + 1 + \phi$ has only four. Both of these figures have four axes of reflective symmetry, and Scholfield does not ask for less symmetric or less repetitious solutions. Figure 7 also shows the square with sides trisected non-optimally as $1 + \rho + \sigma$, which, remarkably, yields only nine similarity classes. It is unknown whether this is best possible for so little symmetry and no repetition.

Another manifestation of "to add is to multiply" involves area. The areas of the sected squares in Figure 8 (with side lengths ϕ and σ) are dissectible. Since $\phi^2 = \phi + 1$, we find areas of 1 and ϕ inside a ϕ-by-ϕ square — in two different ways. And since $\sigma^2 = \sigma + \rho + 1$, the trisected σ-by-σ square is dissectible into contiguous areas of 1, ρ, and σ — this time in six essentially different ways. An analogous square with side length γ (recall the enneagon, Figure 5) and optimally quadrisected sides has area $\gamma^2 = \gamma + \beta + \alpha + 1$, and can be dissected into these four contiguous areas in 17 different ways.

Figure 8. Area dissections



III. Panels and Quasiperiodicity

The early 20th century saw many efforts to formalize the use of ratios like ϕ for the repetition of similar figures in design. (A short bibliography at the end lists some of them.) These efforts culminated in Le Corbusier's Modulor [2], a scale of segments whose lengths form a geometric progression of powers of ϕ. Another supplementary scale had double or half these lengths. These two scales (or two of the same scale), placed perpendicular to each other, produced an array of rectangles with rampant proportionality. Le Corbusier wanted to realize the ultimate application of "to add is to multiply" for design economy, to produce any number of similar but different-sized objects that pack together with no loss of space. Toward this end he used the additive properties of these rectangles (or "panels") for what he called *panel exercises*: choose three or more panels from the array, and use multiple copies of the panels to tile a given rectangle or square in as many ways as possible. The panel exercise was a brilliant teaching tool, and it is a shame that so few wanted to take the lesson.

Another interesting array of panels (Figure 9) makes use of a new idea in mathematics — *quasiperiodic sequences.* A periodic sequence is generated by repetition of a subsequence. For example, in the sequence *abcbabcbabcb...* the subsequence *abcb* is repeated. A quasiperiodic (QP) sequence, generated by an iterated replacement rule, has repetition, but when or how the repetition occurs is in many ways still a mystery. Using the fact that $\phi^2 = 1 + \phi$, generate a sequence on the characters 1 and ϕ by multiplying them by ϕ over and over. The rule is: 1 becomes ϕ, ϕ becomes 1ϕ. When iterated, this rule generates an infinite sequence. From the initial word "1" we have:

$$1 \to \phi \to 1\phi \to \phi 1\phi \to 1\phi\phi 1\phi \to \phi 1\phi 1\phi\phi 1\phi \to 1\phi\phi 1\phi\phi 1\phi 1\phi\phi 1\phi$$
$$\to \phi 1\phi 1\phi\phi 1\phi 1\phi\phi 1\phi\phi 1\phi 1\phi\phi 1\phi \to 1\phi\phi 1\phi\phi 1\phi 1\phi\phi 1\phi\phi 1\phi 1\phi\phi 1\phi 1\phi\phi 1\phi\phi 1\phi 1\phi\phi 1\phi \to \bullet\bullet\bullet$$

The infinite QP sequence formed in this way is not periodic, and yet *any* subsequence will occur infinitely often! Using the relations (A) we can generate eight different sequences on the characters 1, ρ, σ. Here is one. Begin by applying the following rule, amounting to multiplication by σ: 1 becomes σ, ρ becomes $\rho\sigma$, σ becomes $1\sigma\rho$. Then

$$1 \to \sigma \to 1\sigma\rho \to \sigma 1\sigma\rho\rho\sigma \to 1\sigma\rho\sigma 1\sigma\rho\rho\sigma\rho\sigma 1\sigma\rho \to \bullet\bullet\bullet$$

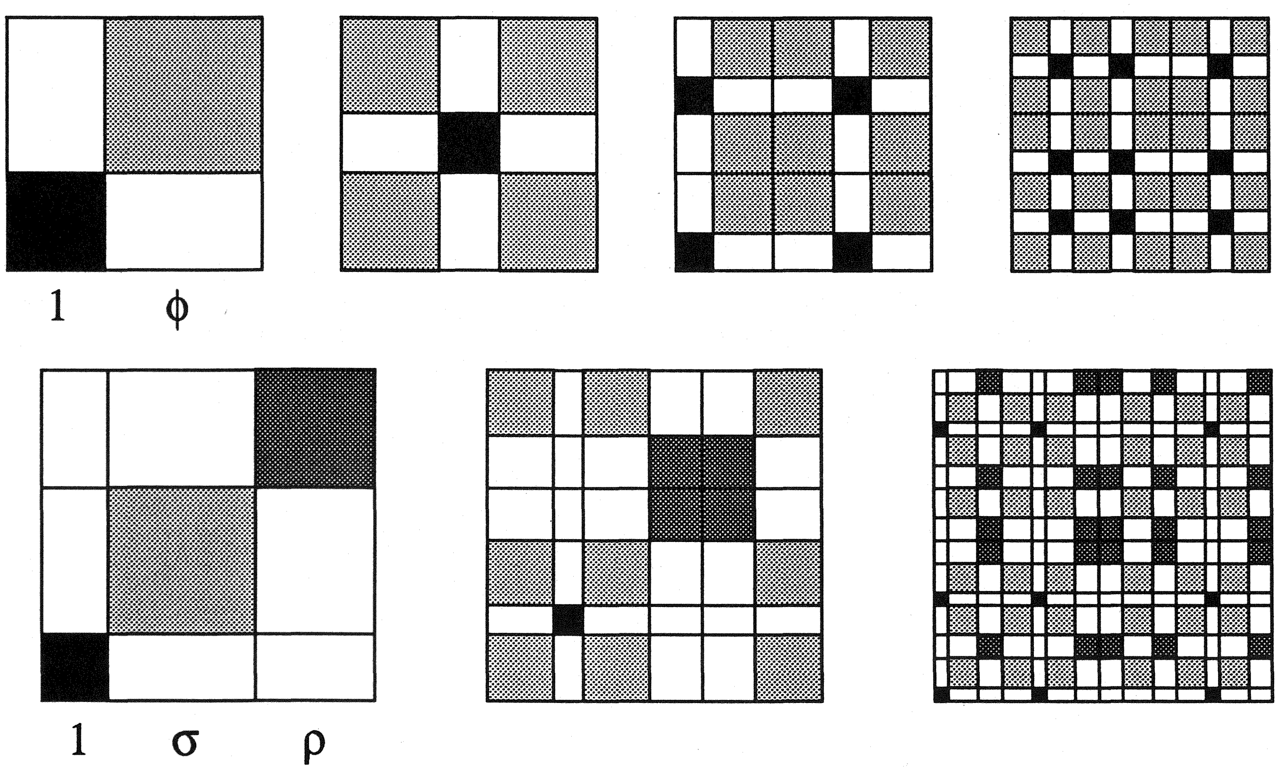
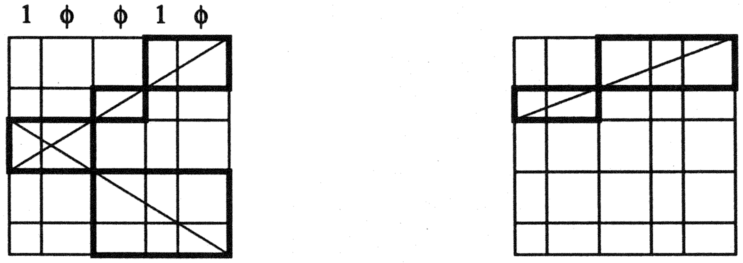


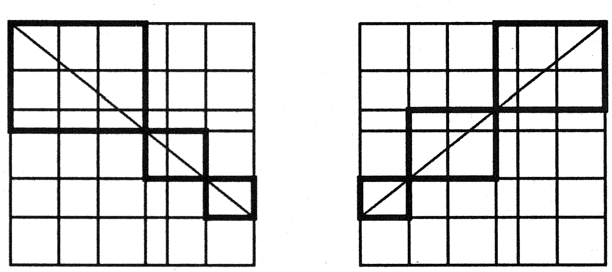
Figure 9. ϕ- and σ-aperiodic tilings

Figure 9 shows the aperiodic tilings created by applying either the ϕ-rule or the σ-rule to the sides of a square. The top sequence employs Modulor panels and could have been drawn by Le Corbusier or Mondrian. These tilings have a profoundly balanced asymmetry and a wealth of proportionality, as revealed in Figure 10. Diagonal regulating lines reveal a few of the similar rectangles and their corresponding proportions. (Figure 10 modifies the heptiling of Figure 9 by using the rule: σ becomes $1\rho\sigma$. This is best possible.) To get an idea of how densely the similarities are packed in Figure 10, consider that a square sected as $\sigma\rho\sigma 1\rho\sigma$ contains 52 rectangles of the similarity class $\rho:\sigma$ alone! This density of similarity is accomplished not only by the ratios' natural tendencies, but also by the QP replacement rules, which spread the three values as evenly as possible, allowing them to interact with each other everywhere. There are analogous constructions for quadrisectional and pentasectional systems.



$$\frac{1}{\phi} = \frac{\phi}{1+\phi} = \frac{1+\phi}{1+2\phi} = \frac{1+2\phi}{2+3\phi} \qquad \frac{1}{1+\phi} = \frac{\phi}{1+2\phi} = \frac{1+\phi}{2+3\phi}$$

$$\frac{1}{\rho} = \frac{\rho}{1+\sigma} = \frac{\sigma}{\rho+\sigma} = \frac{\rho+\sigma}{1+\rho+2\sigma} = \frac{1+\rho}{1+\rho+\sigma} \qquad \frac{1}{\sigma} = \frac{\rho}{\rho+\sigma} = \frac{\sigma}{1+\rho+\sigma} = \frac{1+\rho}{\rho+2\sigma} = \frac{1+\sigma}{1+\rho+2\sigma}$$



$$\frac{\rho}{\sigma} = \frac{\sigma}{1+\rho} = \frac{1+\rho+\sigma}{\rho+2\sigma} = \frac{1+\sigma}{\rho+\sigma} = \frac{\rho+\sigma}{1+\rho+\sigma} = \frac{1+\rho+2\sigma}{1+2\rho+2\sigma} = \frac{1+2\rho+2\sigma}{1+2\rho+3\sigma}$$

Figure 10. Similarities and proportions in the golden and heptagonal systems.

Three-dimensional structures and packings are virtually unexplored. The 1-by-ρ-by-σ box (the pack o' Luckies) is a very pleasant and useful shape, and the 1-by-ρ-by-σ ellipsoid looks like every cobblestone on every beach. Have we overlooked ρ and σ in Nature?

IV. Rational Approximants

The Fibonacci sequence (0,1,1,2,3,5,8,13,21,. . .) is connected to the golden ratio in three senses: (1) Powers of ϕ are expressible as linear combinations with Fibonacci numbers —

$$\begin{aligned}
\phi &= \phi + 0\\
\phi^2 &= \phi + 1\\
\phi^3 &= 2\phi + 1\\
\phi^4 &= 3\phi + 2\\
\phi^5 &= 5\phi + 3 \text{—}
\end{aligned}$$

(2) the right sides of these equations show that each power is the sum of the previous two: $\phi^{n-1} + \phi^n = \phi^{n+1}$; and (3) ratios of consecutive Fibonacci numbers approach ϕ as a limit:

$$1/1,\ 2/1,\ 3/2,\ 5/3,\ 8/5,\ 13/8, \ldots \rightarrow \phi$$

Similarly, we can write linear combinations for powers of σ —

$$\begin{aligned}
\sigma &= 1\sigma + 0 + 0\\
\sigma^2 &= 1\sigma + 1\rho + 1\\
\sigma^3 &= 3\sigma + 2\rho + 1\\
\sigma^4 &= 6\sigma + 5\rho + 3\\
\sigma^5 &= 14\sigma + 11\rho + 6\\
\sigma^6 &= 31\sigma + 25\rho + 14 \text{—}
\end{aligned}$$

and the ratios of coefficients (e.g. 31:25:14) approach σ:ρ:1 as a limit. Better yet, arrange powers of ρ and σ in a 2-dimensional array.

$\rho^{-2}\sigma^5$	$\rho^{-1}\sigma^5$	σ^5	$\rho\sigma^5$	$\rho^2\sigma^5$	$\rho^3\sigma^5$	$\rho^4\sigma^5$
$\rho^{-2}\sigma^4$	$\rho^{-1}\sigma^4$	σ^4	$\rho\sigma^4$	$\rho^2\sigma^4$	$\rho^3\sigma^4$	$\rho^4\sigma^4$
$\rho^{-2}\sigma^3$	$\rho^{-1}\sigma^3$	σ^3	$\rho\sigma^3$	$\rho^2\sigma^3$	$\rho^3\sigma^3$	$\rho^4\sigma^3$
$\rho^{-2}\sigma^2$	$\rho^{-1}\sigma^2$	σ^2	$\rho\sigma^2$	$\rho^2\sigma^2$	$\rho^3\sigma^2$	$\rho^4\sigma^2$
$\rho^{-2}\sigma$	$\rho^{-1}\sigma$	σ	$\rho\sigma$	$\rho^2\sigma$	$\rho^3\sigma$	$\rho^4\sigma$
ρ^{-2}	ρ^{-1}	1	ρ	ρ^2	ρ^3	ρ^4
$\rho^{-2}\sigma^{-1}$	$\rho^{-1}\sigma^{-1}$	σ^{-1}	$\rho\sigma^{-1}$	$\rho^2\sigma^{-1}$	$\rho^3\sigma^{-1}$	$\rho^4\sigma^{-1}$
$\rho^{-2}\sigma^{-2}$	$\rho^{-1}\sigma^{-2}$	σ^{-2}	$\rho\sigma^{-2}$	$\rho^2\sigma^{-2}$	$\rho^3\sigma^{-2}$	$\rho^4\sigma^{-2}$

Then write their linear combinations $a\sigma + b\rho + c$ simply as $a\ b\ c$ in the array below:

5 3 1	8 6 3	**14 11 6**	25 20 11	45 36 20	81 65 36	146 117 65
1 2 2	3 3 2	**6 5 3**	11 9 5	20 16 9	36 29 16	65 52 29
2 0 -1	2 1 0	**3 2 1**	5 4 2	9 7 4	16 13 7	29 23 13
-1 1 2	0 1 1	**1 1 1**	2 2 1	4 3 2	7 6 3	13 10 6
2 -1 -2	1 0 -1	**1 0 0**	1 1 0	2 1 1	3 3 1	6 4 3
-2 1 3	**-1 1 1**	**0 0 1**	**0 1 0**	**1 0 1**	**1 2 0**	**3 1 2**
3 -2 -3	1 0 -2	**1 -1 0**	0 1 -1	1 -1 1	0 2 -1	2 -1 2
-3 1 5	-2 2 1	**0 -1 2**	-1 2 -1	1 -2 2	-1 3 -2	2 -3 3

For example, one reads on the two arrays that $\rho^2\sigma^5 = 45\sigma + 36\rho + 20$. Now notice two remarkable phenomena.

First, 45 36 20 = 20 16 9 (below) + 25 20 11 (left). In fact, each entry in this array of integer triples is the coordinate sum of the entry below and the entry to the left — that is,

$$\rho^m\sigma^n = \rho^{m-1}\sigma^n + \rho^m\sigma^{n-1}.$$

Second, 45:36:20 is an excellent approximation to σ:ρ:1, and the better approximants are farther right and/or up in the array. But where? Some directions yield better improvements than others. For instance, the triple 45:36:20 is a better approximant than any other triple up to its radius from the origin (0 0 1) of the array. Considering these triples as the lattice points of a grid, the best approximants *seem* to lie near a line through the origin with slope $1+\rho$. This has not been explained or proved.

A more concise arrangement of all these integer sequences is given by the *golden matrices.* The relation

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ a+b \end{bmatrix}$$

defines a transformation $(a, b) \to (b, a+b)$ that approaches the ratio 1:ϕ for non-negative values of a and b. Powers of the key matrix —

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 5 \\ 5 & 8 \end{bmatrix}, \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix}, \begin{bmatrix} 8 & 13 \\ 13 & 21 \end{bmatrix}, \ldots$$

— are Fibonacci approximants to the golden proportion $\begin{bmatrix} 1: & \phi \\ =\phi: & 1+\phi \end{bmatrix}$. The relation

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} c \\ b+c \\ a+b+c \end{bmatrix}$$

defines a transformation $(a, b, c) \to (c, b+c, a+b+c)$ that approaches the ratio 1:ρ:σ for non-negative values of a, b, and c. Powers of the key matrix —

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 3 & 5 & 6 \\ 5 & 9 & 11 \\ 6 & 11 & 14 \end{bmatrix}, \begin{bmatrix} 6 & 11 & 14 \\ 11 & 20 & 25 \\ 14 & 25 & 31 \end{bmatrix}, \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}, \ldots$$

— are (3rd-order) Fibonacci approximants to the proportion $\begin{bmatrix} 1: & \rho & : & \sigma \\ =\rho: & 1+\sigma & : & \rho+\sigma \\ =\sigma: & \rho+\sigma & : & 1+\rho+\sigma \end{bmatrix}$.

Numbers in the array at the bottom of the previous page occur either in these last matrices or as sums of matrix entries. This second sequence of matrices contain what are now called the "3rd-order Fibonacci numbers." The 4th- and 5th-order Fibonacci numbers — approximating the diagonals of the 9-gon and 11-gon — are obtained by extending the 0/1 key matrices above to 4×4 and 5×5. *Mathematica* will give you a page full of matrices for the following line. Adjust for larger matrices.

```
Table[MatrixForm[MatrixPower[{{0,0,1},{0,1,1},{1,1,1}}, n]], {n,1,15}]
```

After Scholfield surveyed the history of proportion from Vitruvius to Le Corbusier, the art world lost interest in the subject. Today it is often said that no one *designs* anymore — they just *express* themselves. Twenty years after Scholfield's survey, quasi-crystals were discovered (see Grünbaum & Shephard), and mathematicians, who had thought that everything meaningful had been said about proportion centuries ago, suddenly reopened the subject. In their efforts to explain the new phenomena they took a new and closer look at the golden ratio and found themselves asking questions that designers had asked and forgotten. Le Corbusier and others of his generation wanted to know whether the golden "key to the door of the miracle of numbers" was a unique phenomenon, and perhaps someone is still waiting for an answer. The mathematician's answer is emphatically *no*; there is an infinitude of irrational numbers whose geometric sequences of powers have additive properties useful for the repetition of similar figures. Crystallographers have recently named the silver, copper, and bronze ratios, among others. And there are three new and unapplied patterns belonging to the golden family — the optimal trisection, quadrisection, and pentasection — that concentrate maximal repetition of ratio into the least space. The possibilities await our exploration.

References

[1] E. J. Dijksterhuis, *Archimedes*. Copenhagen: Ejnar Munksgaard, 1956.

[2] Le Corbusier, *The Modulor*. Cambridge: Harvard University Press, 1954.

[3] Robert Lawlor, *Sacred Geometry*. London: Thames & Hudson, 1992.

[4] P. H. Scholfield, *The Theory of Proportion in Architecture*. Cambridge: The University Press, 1958.

[5] Peter Steinbach, *Golden Fields: A Case for the Heptagon*. MAA Mathematics Magazine, February 1997.

[6] Kim Williams, *The Sacred Cut Revisited: The Pavement of the Baptistery of San Giovanni, Florence*. The Mathematical Intelligencer 16:2 (1994) 18.

Related Reading

György Doczi, *The Power of Limits*. Boulder: Shambala, 1981.

Matila Ghyka, *The Geometry of Art and Life*. New York: Dover, 1978.

Grünbaum & Shephard, *Tilings and Patterns*. New York: Freeman and Co., 1987.

Jonathan Hale, *The Old Way of Seeing*. New York: Houghton Mifflin, 1994.

Jay Hambidge, *The Elements of Dynamic Symmetry*. New Haven, 1948.

Gyorgy Kepes, *Module, Proportion, Symmetry, Rhythm*. New York: George Braziller, 1966.

Nelson, Joseph, & Williams, *Multicultural Mathematics*. Oxford University Press, 1993.

Dan Pedoe, *Geometry and the Visual Arts*. New York: Dover, 1976.