BRIDGES
Mathematical Connections
in Art, Music, and Science

A Topology for Figural Ambiguity

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Abstract

A topology for figural ambiguity (braid theory) is proposed which describes figure-figure, figure-ground, and figure-ground-figure ambiguities with the defining relations of the topology. The power of the topology is demonstrated by the classification of ambiguities and the revelation of new ambiguous forms.

Introduction

Ambiguous figures, configurations that have more than one perceptual interpretation (Figure 1), have wide appeal; they are often regarded with surprise, amazement, curiosity, bewilderment, and even amusement.

Figure 1. *Agule's ambiguity*[1]. *Is this man a truth teller or a liar. The answer is written all over his face (rotate 1/3 counterclockwise).*

They have been around since antiquity, but only recently have there been attempts at theoretical explanations of these enigmatic patterns. Any theoretical account of them should, at a minimum, incorporate the following in its narrative:

1. *Stimulus Constancy:* The stimulus pattern does not change.
2. *Stimulus Segmentation:* The stimulus pattern is either partitioned (e.g., figure-ground) or it is not (e.g., figure-figure).
3. *Response Multiplicity:* There is more than one interpretation of the stimulus pattern.
4. *Response Saliency:* One and only one interpretation of the stimulus pattern can be attended to at any given moment.

The logical result of 3 and 4 is *Response Abruptness:* The change in salience is sudden and complete–all or none.

Topology, a mathematics that concerns itself with invariance under conditions of continuous distortion or change (*homotopic* equivalence), addresses both stimulus constancy and response multiplicity.

To use an example that has almost become a cliché, a torus can be smoothly and continuously transformed into a cup shape, but, topologically, the cup remains, fundamentally, a torus; thus a topologist cannot tell the difference between the doughnut he eats and the cup from which he drinks his coffee. The ambiguous stimulus does not change, but the multiple perceptual interpretations of it are as different as the coffee cup and the doughnut.

Algebraic Topology and the Theory of Braids

The topology of braids [2] is an algebraic topology particularly appropriate for the purposes at hand, since it appears helpful in describing "impossible" figures, seen in the artistry of Oscar Reutersvaard and M. C. Escher [3,4], and perceptual impossibilities and ambiguities are likely related [5].

Consider a finite number of strings stretched between two frames g_1 and g_2 (Figure 2a). If there are

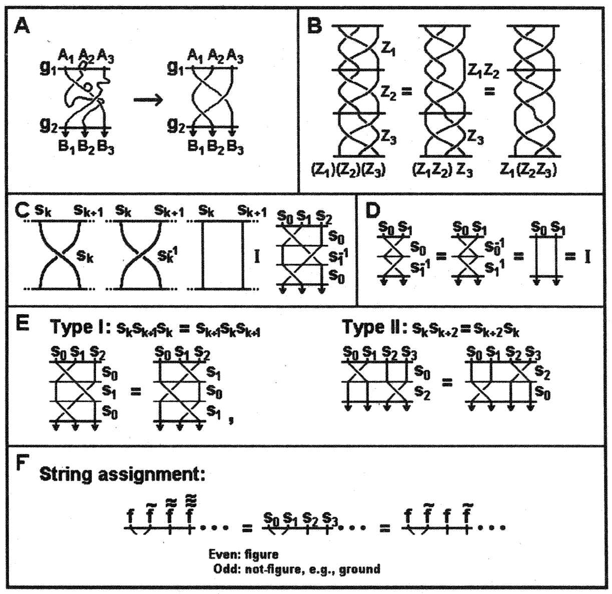

Figure 2. *The topology of braids. (a) An unacceptable and acceptable braid. (b) The associativity of briads. (c) Right. A pigtail braid partitioned into single nodes. The other figures represent the three possible forms of a single node, s_n, s_n^{-1}, I. (d) Inverse nodes. A tug on the frames creates an identity braid. (e) Type I and Type II relations. These are topological equivalences (homotopies) that help define the algebra of the topology. (e) String assignment for the ambiguity problem (see text for description).*

n strings, then g_1 and g_2 are divided into segments by points A_1 A_2 A_3 ..... A_n and B_1 B_2 B_3..... B_n. The points are ordered from left to right. The strings can cross, but no string crosses the frame, and the courses of the strings are such that any line parallel to g_1 and g_2 crosses each string exactly once. Thus the loops and whorls in any of the lines shown in Figure 2a, left, are not allowed; neither is the frame crossing of A_2. A crossing of more than two strings at a single node (Figure 2a, left) is also unacceptable. The strings are moved to produce a distribution of crossings such that any line parallel to g_1 and g_2 never passes through more than one crossing. The order of the crossings is read from g_1 to g_2 (Figure 2a, right).

If two braids Z_i and Z_j have an equal number of strings, they can be combined by placing the upper frame of one against the lower frame of the other then suppressing the two overlapped frames. Such a binary operation is called a concatenation (product) of Z_i and Z_j and is expressed by $Z_iZ_j = Z_k$. It is obvious that this operation is associative. That is, $(Z_1Z_2)Z_3 = Z_1(Z_2Z_3)$ (Figure 2b). The commutative law does not hold in general: $Z_iZ_j \neq Z_j Z_i$.

Since no two crossings lie on any one line parallel to g_1 and g_2, any braid, no matter how complex, can be divided into a series of n connected braids (Figure 2c, right), each with one of the following properties: (1) The kth string from the left crosses over the k + 1st string. This operation is called s_k and is shown in Figure 2c, left. (2) The kth string from the left passes under the k + 1st string. This operation is labeled s_k^{-1} and is shown in Fig. 2c, second from the left. (3) There are no crossings (Fig. 2c, second from the right). This braid is denoted by I. It is a unit braid or identity braid since ZI = IZ = Z; the joining of I to any braid does not alter its structure. It is clear that s_k and s_k^{-1} are inverse operations since $s_k s_k^{-1} = s_k^{-1} s_k = \mathrm{I}$ (Figure 2d). (Just as s_k has an inverse, so does the more complex braid, Z. The inverse of Z, Z^{-1}, is a mirror image of Z.) This notation allows the braids to be treated symbolically. For example, the common pigtail braid shown in Figures 2a and 2c, right, is given by $s_0 s_1^{-1} s_0$. The two braids in Figure 2e describe two relations, $s_0 s_2 = s_2 s_0$ and $s_0 s_1 s_0 = s_1 s_0 s_1$, that help define the algebra, and they also play a central role in the description of figure-ground ambiguity.

Braid Rank and File Assignments for Figure Ambiguity. The vertically ordered rows within the braid are given position numbers, and the horizontal string positions (Figure 2f) are labeled $f\tilde{f}\tilde{\tilde{f}}\cdots$ so that a count is imposed on the strings by the number of negations (~). This count defines the string indices. The sequence of events $f\tilde{f}\tilde{\tilde{f}}\cdots$, of course, is equivalent to $f\tilde{f}f\cdots$. The only assignments imposed on the braid are the status of a string at the node (s_k, s_k^{-1}), the vertically arranged horizontal sections created by the separation of nodes into one node per section, and two mutually exclusive entities, s_{even} and s_{odd} or f and $\tilde{f}$.

Neither the labeling scheme nor the topology makes any assumptions about perception, but the assigned properties can serve as a template onto which perceptual primitives can be mapped. For example, f or s_{even} is read as figure, and $\tilde{f}$ or s_{odd} as non-figure, commonly regarded as ground in figure-ground problems $(s_0 s_1 s_2 \cdots \equiv fgf\cdots)$. Figure and ground, of course, are mutually exclusive. In the case of figure-figure ambiguities, f and $\tilde{f}$ represent two mutually exclusive interpretations. Left string dominance, imposed by a positive exponent (s_k), represents perceptual salience (e.g., *f*) and its inverse (s_k^{-1}) perceptual non-salience (e.g., ~*f*). Finally, the vertical braid segments with their internal frames are logical assignments of left to right figure segments with their partitions.

The assumptions concerning perception are clearly minimal, viz., the existence of mutually exclusive perceptual interpretations (e.g., figure-ground separation or horizontal string assignments), salience (nodes), and partitioned segments (node separation). The complexities of ambiguity are derived from the topology. These offer a unique look at perceptual interpretation and salience interplay found in figural ambiguity.

Simple Ambiguities

Three segment figures. The face-vase problem appears in Figure 3a, left. Reading from left to right the observer encounters a face-ground-face or ground-vase-ground perceptual organization. During the perceptual switch left *face* passes to left *ground*, center *ground* to center *vase*, and right *face* to right *ground.*

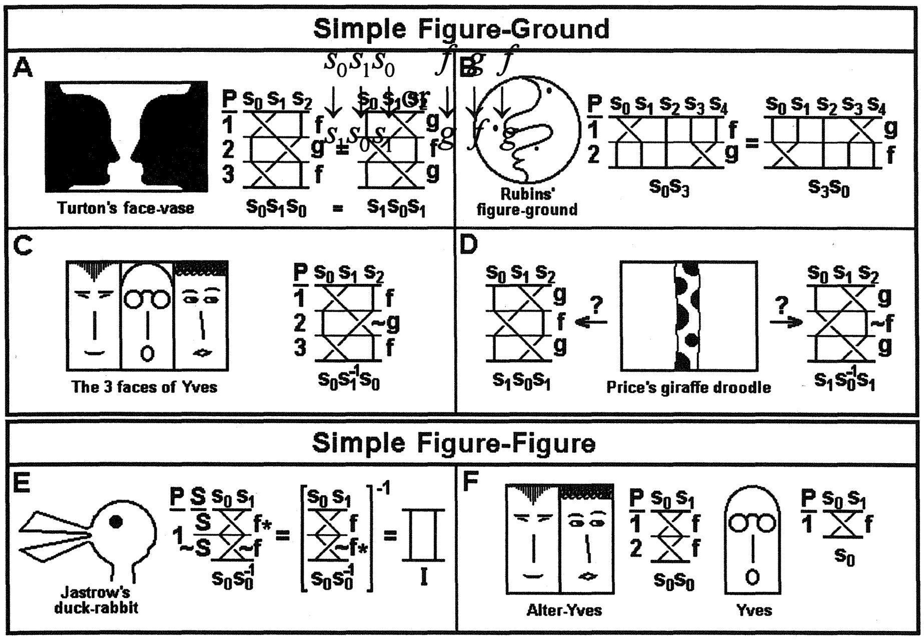

Figure 3. Simple ambiguities. (a) Three segment figure-ground ambiguity. (b) Two segment figure-ground ambiguity. (c) Stable figure-figure configuration. (d) Roger Price's giraffe droodle. (f) Figure-figure ambiguity with its inverse representation. (e) Representations of stable figure-figure configurations.

Everything that passes is *contra*; that is, there is consistency in the type of transition for all segments, which is something one would expect in a *global* transformation where everything changes at the same time.

$$\begin{array}{c} s_0 s_1 s_0 \\ \downarrow\downarrow\downarrow \\ s_1 s_0 s_1 \end{array} \text{ or } \begin{array}{c} f\,g\,f \\ \downarrow\downarrow\downarrow \\ g\,f\,g \end{array}.$$

The Type I relation seems to represent this ambiguity:

$$s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1}\text{: } s_0 s_1 s_0 = s_1 s_0 s_1 \text{ or } fgf = gfg. \qquad (1)$$

The switch is reflected in the homotopic shift between the two twisted forms (*multiplicity*), and this topological equivalence implies they are taken from an unchanging stimulus (*constancy*). Two internal frames are fixed implying two fixed partitions within the figure (*segmentation*). Topological equivalence is not assumed but is revealed directly by the homotopic shift of the relation strings. The string flexion effecting this string movement is smooth and continuous, but shifting node location is not; it is sudden and discrete like the perceptual shift observed in figural ambiguity (*abruptness*). Finally, all nodes are overpasses (*salience*).

Two segment figures. A figure-ground ambiguity appears in Figure 3b. Reading from left to right, a face is encountered, then its background. Suddenly one sees the face on the right with its background on the left. It is difficult to see both faces at the same time. The left *face* passes to left *ground*, and right *ground* passes to right *face*

$$\begin{array}{c} s_{even} s_{odd} \\ \downarrow\ \downarrow \\ s_{odd} s_{even} \end{array} \text{ or } \begin{array}{c} f\ g \\ \downarrow\ \downarrow \\ g\ f \end{array}.$$

Again, everything is *contra* supporting a global shift in perception. A Type II relation with nodes s_0s_3, their even (f) and odd (g) indices appropriately separated for positive global saliency, correctly describes alternating, salient perceptual states:

$$s_k s_{k+i+2} = s_{k+i+2} s_k: \; s_0 s_3 = s_3 s_0 \text{ or } fg = gf. \tag{1}$$

The extended separation of the nodes is necessary since s_0s_2 is figure-figure and $s_0s_2^{-1}$ is figure-not-figure ($f \sim f$ not $f \; \tilde{f}$) which disallows global salience. That is, figure-not-figure is a local description and figure-ground is global.

Again, topological equivalence is not merely assumed; it is a consequence of node shift, which implies an unchanging stimulus (*constancy*) in the presence of changing perceptual interpretation (*multiplicity*). Again, the internal fame is fixed indicating a permanent partition in the figure-ground pattern (*segmentation*). As before, all nodes are overpasses (*salience*).

Cognition: Braids say little about cognition, "the ghost in the machine," and this is part of the appeal of the representation. While braids say nothing about what the cognitive process is, beyond salience and figure-ground separation, they suggest where cognition lies.

First, in the case of *local* salience, piecemeal perception, the appropriate braids are simple pig-tail braids (Figure 3c) with immutable patterns, and they possess fixed internal frames implying fixed partitions ($s_1s_0^{-1}s_1$). Now consider Figure 3d, one of Roger Price's "droodles" [6], a giraffe passing by a second story window, a seemingly stable figure if there ever was one. However, the stable pigtail braid $s_1s_0^{-1}s_1$ ($g \sim f g$) does not describe the figure, but the flexible relation, $s_1s_0s_1$ or gfg, does. Yet, the droodle does not appear ambiguous, and the flexible string is perplexing.

However, the topology does not demand that the string be flexed–that the homotopy be manifested; it only provides the possibility, and the assumption that the middle string in $s_1s_0s_1$ can show resistance from full flex to rigidity seems viable. The process responsible for this resistance (or lack of it) is cognition.

For a more direct demonstrable link between cognitive influence and relation shift, an attempt will be made to influence the reader's perceptual predisposition: Price's giraffe is not a giraffe at all but a view from a basement apartment window, through parted curtains, of a leopard giving chase. This transforms the droodle into an ambiguity represented by the Type I relation braid with an adjustable middle string that formerly resisted flexing.

One segment figures: Figure-figure ambiguities, like their figure-ground cousins, reveal two interpretations that are independent and mutually exclusive; when one is seen the other is not. We begin by concatenating salience and non-salience forming two sections of a single braid, which represent overt and covert percepts. The frame between sections (|) divides overt and covert perceptual responses and replaces the segment partition seen in figure-ground patterns. The braid forms $[s_0 | s_0^{-1}]$ or $[f | \sim f]$ appear next to Jastrow's ambiguity in Figure 3e.

Rabbit has been assigned to *f*. If Rabbit is salient, Duck is non-salient and assigned to *~f*. Rabbit is distinguished from Duck by *; when * is encountered, read "Rabbit." How is Duck revealed? There are three braid functions available for the ambiguity problem: Type I and Type II relations and the inverse. Since the two relations have already been spoken for, we are forced to choose the inverse. Braid inversion is effected by appropriately distributing the negative exponent then inverting the order of the terms. This moves the hidden duck and its perceptual non-salience to the top of the braid where the sign for non-salience is lost from the duck, and gained by the rabbit. Here figure passes to figure, i.e.,

$$\begin{array}{ccc} s_{0*}s_0^{-1} & & f \sim f \\ \downarrow\downarrow & \text{or} & \downarrow\ \downarrow \\ s_0 s_{0*}^{-1} & & f \sim f \end{array} .$$

Everything that passes is *ipsi*. The fact that braid forms $s_{0*}s_0^{-1}$ and $s_0 s_{0*}^{-1}$ are topologically equivalent, stimulus *constancy* under conditions of a changing perceptual interpretation is observed (*multiplicity*).

As for "Alter Yves," $s_0 s_0$ (ff) captures the essence of two equally salient faces. Furthermore, there is one internal frame which matches the internal partition separating the faces. Yves, a figure without partition, is represented by a single node with no internal frame. If internal frames represent internal partitions what is to be done with the internal frame of $s_{0*}s_0^{-1}$ representing Jastrow's figure, a figure without partition? The braid group provides an elegant solution to the *partition* problem. The two braids for the two interpretations each reduces to an identity braid in the manner of all inverse pairs,

$$[s_{0*}s_0^{-1}]^{-1} = s_0 s_{0*}^{-1} = \mathbf{I}\,, \tag{3}$$

and identities have no internal frames, hence no partitions are present in the figure..

Complex Ambiguities

A complex ambiguity is composed of simple ambiguities. Figure 4a is a figure-figure ambiguity, two profiles or two halves of a full face. When one observes a vase in Turton's figure (Figure 3a), ground is seen either as two flanking regions or a single background passing behind the vase by the Gestalt principle of good continuation; here there is no such choice.

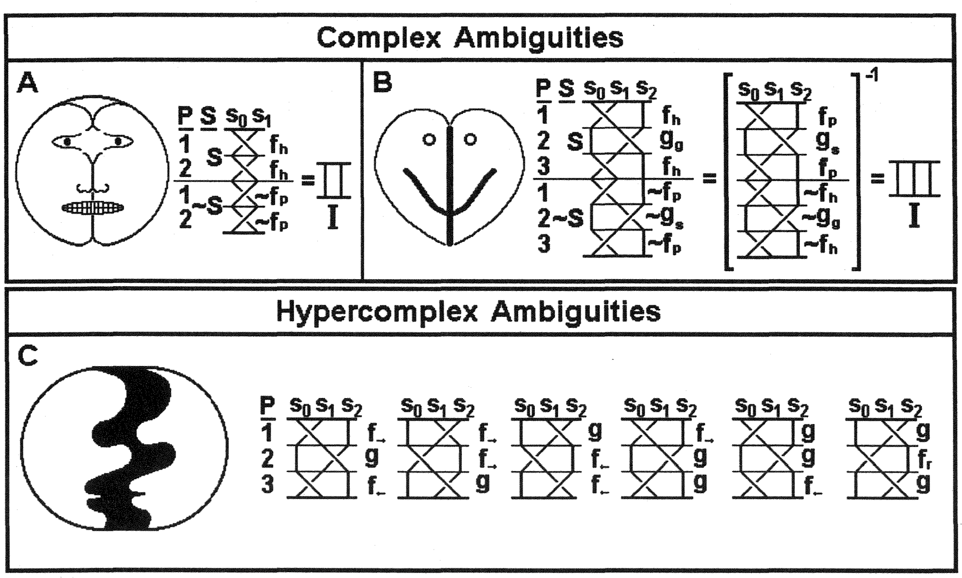

Figure 4. Complex and hypercomplex ambiguities. (a) Complex figure-figure ambiguity. (b) Complex figure-ground-figure ambiguity with figure-figure components. (c) Hypercomplex figure-ground-figure, figure-figure-ground, and figure-ground-ground ambiguity with their homotopic braid forms. This figure possesses fifteen ambiguities.

There are three interpretations of Figure 4b. Taking things segment by segment: profile, grins +

separation, profile; half face, full smile, half face, and ground, psi, ground. The number of ambiguities equals the number of interpretations taken two at a time giving three ambiguities. These include two profiles and two halves of a full face each perceptually switching (homotopically) with the Greek letter psi. That is, from left to right, *figure* passes to *ground*, *ground* passes to *figure*, and *figure* passes to *ground* again,

$$\begin{array}{ccc} f & g & f \\ \downarrow & \downarrow & \downarrow \\ g & f & g \end{array} .$$

Everything that passes is *contra*. Each ambiguity in relation form,

$$s_0 s_1 s_0 = s_1 s_0 s_1 : fgf = gfg . \qquad (9)$$

The third ambiguity is figure-figure consisting of two *half faces* and a full *smile* passing to two *profiles* and a pair of *grins* as seen in Figure 5a in part. That is,

$$\begin{array}{ccc} f & g & f \\ \downarrow & \downarrow & \downarrow \\ f & g & f \end{array} \text{ or } \begin{array}{ccc} h & s & h \\ \downarrow & \downarrow & \downarrow \\ p & g & p \end{array} . \qquad (10)$$

Everything that passes is *ipsi* (same). In terms of braids,

$$s_{0h} s_{1s} s_{0h} | s_{0p}^{-1} s_{1g}^{-1} s_{0p}^{-1} = s_{0p} s_{1g} s_{0p} | s_{0h}^{-1} s_{1s}^{-1} s_{0h}^{-1} . \qquad (11)$$

The expression for the complete set of ambiguities would be

$$s_{0h} s_{1s} s_{0h} | s_{0p}^{-1} s_{1g}^{-1} s_{0p}^{-1} | s_{1g} s_{0\psi} \, s_{1g} .$$

(Changing to the homotopy of both facial forms, $s_{1g} s_{0\psi} s_{1g} | s_{1g}^{-1} s_{0\psi}^{-1} s_{1g}^{-1}$, produces a vacuous figure-figure ambiguity at best.)

Hypercomplex Ambiguities

A crucial test of a model is whether or not it can create something new–here a different ambiguous form. Consider the ambiguity in Figure 4c and its interpretation of two faces looking right with ground at the right (ffg). The braid $s_0 s_0 s_1$ cannot represent this since it is inflexible, and we need at least two homotopic forms to represent two faces looking left (gff) and two faces looking at each other (fgf).

A variant of the Type II relation is $s_k s_{k+1} s_k^{-1} = s_{k+1}^{-1} s_k s_{k+1}$, a braid whose middle string passes under the other two strings rather than between them. That is,

$$s_1^{-1} s_0 s_1 = s_0 s_1 s_0^{-1} \text{ or } \sim gfg = fg \sim f$$

an odd mix of figure-ground and salience and non-salience. If we interpret this as

$$s_1^{-1} s_0 s_1 = s_0 s_1 s_0^{-1} \text{ or } ffg = fgg$$

we find two faces looking left are homotopic to one face looking left toward two grounds. Thus a new ambiguity is revealed: The middle portion can be seen as ground (shadow?) to the left-most face or a figure, a face looking left with its ground to the right. This is a weak ambiguity, more interpretive than perceptual, but, nonetheless, it is an ambiguity.

Thus far there are two ambiguities which serve as two figure-figure composites. That is,

$$s_1^{-1} s_0 s_1 | s_1 s_0 s_1^{-1}$$

or ffg and its homotopy fgg in a figure-figure relationship with gff and its homotopy ggf. Add to this two faces looking at each other (fgf) with its paired interpretation, a wire loop with a black ribbon attached across the loop in the middle (gfg) or $s_1 s_0 s_1 = s_0 s_1 s_0$. This gives the final expression,

$$s_1^{-1} s_0 s_1 | s_1 s_0 s_1^{-1} | s_0 s_1 s_0 .$$

There are six interpretations here (see Figure 4c), and this seemingly simple looking figure boasts 15 different ambiguities.

Conclusions

Topology is proposed as a logical mathematical approach to figural ambiguity. It describes the perceptual *experience* (phenomenology) of figural ambiguity rather than cognitive influence. The topology of braids as a model for ambiguity meets all the criteria set forth at the beginning of this paper: It reflects stimulus constancy through topological equivalence. It satisfies stimulus segmentation through fixed internal frames and, in the case of partitionless figure-figure ambiguities, a reduction to **I**. Response multiplicity is revealed by the relations and the inverse operation. Saliency is represented by string dominance at the nodes. Finally, response abruptness follows from the braid rule that no two nodes occupy the same segment during a homotopic shift.

Braid topology reveals characteristics of ambiguities that have been ignored: It suggests complex and hypercomplex ambiguities and relates these to simple ambiguities thereby creating a classification that, heretofore, has not been seen. Finally, it points the way to new ambiguous forms.

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