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Mathematical Connections
in Art, Music, and Science

The Circle: A Paradigm for Paradox

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Abstract

The ultimate desire of mankind is to identify wholeness, to grasp the essence of being, to be integrated with the harmony, perfection, patterns, and cycles of the material, metaphorical and metaphysical worlds. This desire motivates us to explore the realms of fact and fancy, logic and metaphor, reason and emotion, to capture the whole of being in one part, to see it, hear it, feel it, and enjoy it in everyday life.

The circle is an object of nature, an idealization of pure mathematics, and a symbol or framework we use to understand and describe our world. The circle exists independently of human thought, as ripples in a pond, or the appearance of the sun and moon, or the shape of the iris of an eye. In mathematics, we choose to define a circle as the places at a constant distance from a center, usually in two dimensions. In this article, we look back at world history and the varied uses of the circle: literal and literary, physical and poetical, mathematical, metaphorical and mystical.

Ancient Beliefs

The circle crowned the head of the ancient Egyptians' sun god, *Ra*. The Celtic Druids carved circular and spiral patterns in stone monuments. Their sacred places had a circle of large stones up to thirty yards in diameter, such as at Stonehenge. The *Yin-Yang* symbol of two parts spiraling within a circle is a traditional icon of Confucianism and Taoism (figure 1). It suggests movement around the inside of the circle. It also provides a paradigm of polarity with which to view the dynamics of everyday life. As a symbol, it can be as personal and internal as a heart, which gives and receives blood through each complete cycle. It can also be as general and external as the cycles of day and night.

The Buddhist circular mandala designs have been used continuously for millennia. "A mandala (Sanskrit for "circle") is a symbolic diagram of the universe, arranged in circles, used in tantric Buddhism. The Swiss psychologist Carl Jung considered the mandala to be a universally occurring pattern associated with the mythological representation of the self [1]."

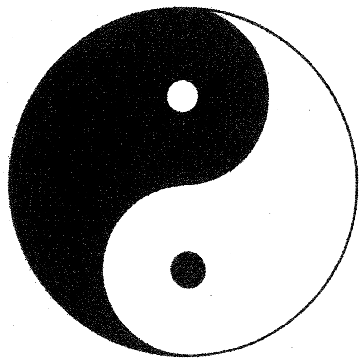

Figure 1. The *yin-yang* symbol of Confucianism and Taoism.

Zoroastrianism, the tradition of ancient Persia, is believed by scholars to have been influential in the later development of metaphysical concepts in Abrahamic and Eastern religious beliefs [2a]. Its influence survives to our own day, not only in central Asia, but in such products of the European post-Romantic movement as Richard Strauss's music and Friedrich Nietzsche's book, *Thus Spoke Zarathustra*. Modern historians have dated the time of Zoroaster to approximately 1750 BC. Also known as Zarathustra, he was the founder of the Zoroastrian tradition [2b].

One symbol of Zoroastrianism is the *Fra-vahar*, a figure which stands for the ideal moral and spiritual focus in life (figure 2). *Fra* is the direction, forward, and *vahar* describes a pulling force. Of the two circles in the figure, the ring in the hand is a reminder that we are bound to keep our promises or agreements with others. The other circle, at the waist, reminds us that our spirits live on, in essence immortal, and so also symbolizes infinity.

Figure 2. The *Fra-vahar* symbol of Zoroastrianism.

A series of coaxial circles is used on the Great Stupa temple at Sanchi, India, to indicate higher levels of being [3]. The Hindu cosmology included spheres as well as circles. The Navajo *kiva* or prayer room is circular, as are many mosques [4].

The circle has represented the divine, as well as the universe and any group of people in it [5].

> Isaiah said: "To whom then will ye liken God? or what likeness will ye compare unto him? . . . Have ye not known? have ye not heard? hath it not been told you from the beginning? have ye not understood from the foundations of the earth? It is he that sitteth upon the circle of the earth, and the inhabitants thereof are as grasshoppers; . . ."

A saying derived from words attributed to Empedocles suggests the ancient Greeks may have also thought of these connections [6].

> The nature of God is a circle of which the center is everywhere and the circumference is nowhere.

The Greeks tried to explain the world, from the mystical to the scientific. The Pythagoreans developed geometrical relations for much of the world around them, but found the apparently simple circle to be a serious challenge. The Babylonians, Egyptians, and Greeks recognized that the ratio of a circle's circumference or periphery to its diameter was a constant, found to be about 3.14, which we call π (pi). The Pythagorean Greeks hoped to use geometry to relate the areas of the figures of a circle and a square, or "to square a circle" using a compass and straight edge.

Johann H. Lambert, the great eighteenth-century logician, was the first to demonstrate the irrationality of π. Beyond the classification of real numbers into rational and irrational numbers, the irrationals can also be classified as either algebraic or transcendental. A number is said to be algebraic if it can be the root of a finite polynomial equation with integer coefficients not all zero. A real number that is not algebraic is called transcendental. In 1882, the German mathematician Ferdinand Lindemann finally proved that π is a transcendental number. This settled the question of squaring the circle, since only algebraic dimensions can be constructed by using a compass and straight edge.

The Pythagoreans developed a method to represent the area of a circle by dividing it up into an infinite number of pie slices, which are essentially infinitely narrow triangles, as high as the circle's radius (figure 3) [7]. This gives the area by applying the simple formula for triangles. However, this converts the problem of the circle into the problem of infinity. Zeno confronted the Pythagoreans by asking if a slice of pie with a curved base can be represented with a flat-edged triangle before it is infinitely thin, and if the triangle is infinitely thin, then how can a real area be described by a collection of "nothing." The Greek search for mathematical rigor faced a daunting paradox.

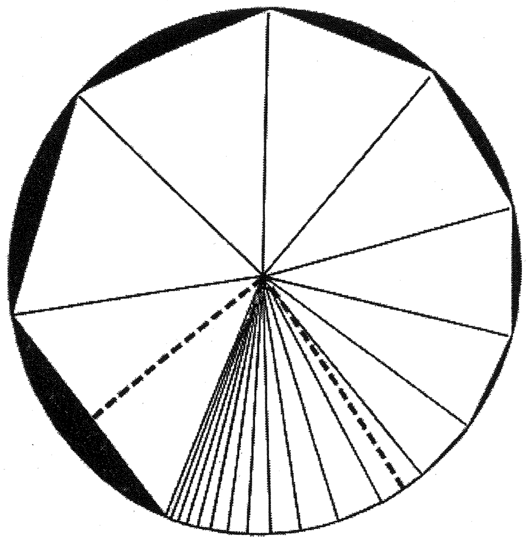

Figure 3. The circle as the sum of an infinite number of small perfect triangles. "The smaller the triangles that are inscribed within a circle, the more nearly they fill it up, shown in the diminishing shaded areas above. Moreover, as these triangles shrink, their height (*dotted line*) gets closer to the length of the radius . . ." [7].

Different cultures throughout history have associated the square with the tangible world, and the circle with the perfect, ideal or the divine universe. The Pythagoreans and Plato associated the five regular *Platonic* solids with the four ancient *elements* of Empedocles, where earth is thought of as the cube, in the sense that one lives within one's own four walls. Similarly, fire is the tetrahedron, and the almost-spherical dodecahedron represents the universe. This is illustrated in figure 4, from Johannes Kepler's *Harmonices Mundi, Book II* (1619) [8].

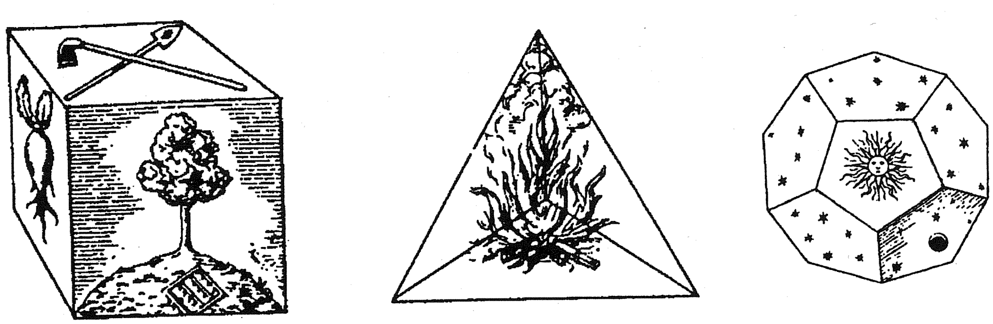

Figure 4. The cube, tetrahedron, and dodecahedron regular solids, representing fire, earth, and the universe, from Kepler's *Harmonices Mundi* [8].

Perhaps because the circle couldn't be squared mathematically, it was intriguing to do it visually. Leonardo DaVinci fit the proportions of the limbs of a human body into a circle and a square (figure 5). Leonardo was illustrating the "ideal" proportions of human limbs relative to body size.

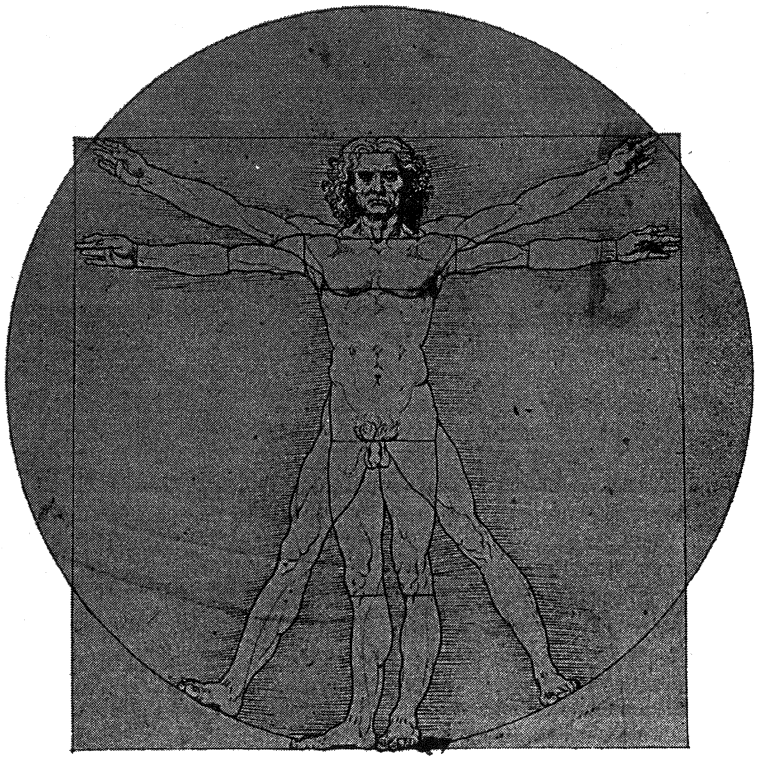

Figure 5. Leonardo DaVinci's man in a circle and a square.

The Scandal of Geometry

The circle is an idealized object, the fundamental ultimate in perfection. In Euclid's geometry, the circle is the only geometrical area that appears in a fundamental postulate. Euclid constructed his geometry based on five postulates. Four of Euclid's five basic postulates seem trivial enough to be considered as axioms. However, the fifth one, the *parallel* postulate, from the very beginning was considered as insufficiently plausible to qualify as an unproved assumption by mathematicians. For two thousand years, mathematicians unsuccessfully tried to derive it from the other four postulates, or to replace it with another more self-evident one.

Euclid himself did not quite trust this postulate. He postponed using it in a proof for as long as possible, until his 29th proposition. A modified version of this postulate (equivalent to the original one) is as follows:

> For every line l and for every point p that does not lie on l there exists a unique line m through p that is parallel to l.

To make Euclid's geometry rigorous, many larger systems of axioms have been proposed. The one proposed by David Hilbert was not the first, but was the closest in spirit to Euclid's. Today, a system of geometry without a form of the parallel postulate is called the *neutral geometry*.

The discovery of non-Euclidean geometry occurred in the nineteenth century with the works of János Bolyai and N. I. Lobachevsky (There is evidence that Gauss also discovered some of the non-Euclidean results, but did not publish them.) One of the most prominent non-Euclidean geometries is Hyperbolic geometry, which comes from the neutral geometry combined with the Hyperbolic Postulate:

> For every line l and point p not on l there exist at least two distinct lines parallel to l that pass through p.

We can prove, in this geometry, that the sum of the angles of any triangle is strictly less than π. Interestingly, a direct relationship between hyperbolic geometry and Einstein's special theory of relativity was discovered by the physicist Arnold Sommerfeld in 1909, and elucidated by the geometer Vladimir Vančak in 1912.

Since a star's gravity bends light toward itself, a triangle formed by light rays between three stars has this geometry (figure 6). So, in some ways, this non-Euclidean geometry is a better description of the real interstellar universe than our traditional Euclidean geometry.

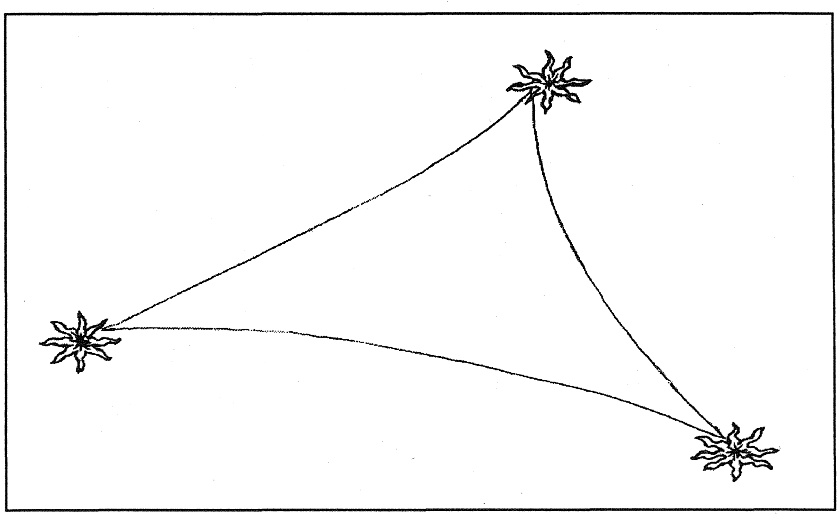

Figure 6. Light rays bend between stars, as do lines in hyperbolic geometry.

To explore hyperbolic geometry, Eugenio Beltrami constructed an explicit model in 1866. This model, now known as the *Klein* model, represents n-dimensional hyperbolic space by an open ball in Euclidean space, and represents hyperbolic lines by Euclidean straight-line segments in the ball.

Another model was invented by Poincaré. His geometric model is *conformal*, which means it preserves angles. The two-dimensional model maps the hyperbolic plane of all space onto a unit-radius size disk. Here, hyperbolic lines are represented by arcs of circles perpendicular to the bounding circle of the unit disk. As everything in this model is inside the unit disk, its circumference represents infinity, and outside the boundary circle there is absolutely nothing.

To understand the Poincaré model, consider that you are in the center of such a circular world, and want to walk away toward the boundary circle at infinity. From your local perspective, each step you take is the same size. But from the point of view of an observer outside of this plane, the first step is followed by successively smaller steps. Your steps get progressively smaller, to the observer's eye, by the ratio of $(1 - r^2)/2$, where r is your distance from the origin in this model [9].

Let us see what we can do in this circular world that we cannot do in our world of apparently Euclidean geometry. Consider that we want to cover a Euclidean surface with some simple type of regular polygon, without any gaps or overlaps. We see that an equilateral triangle can cover a surface, as can a square, or a regular hexagon. But no other regular polygons can do this. The reason is that if p is the number of sides of a regular polygon (call it a p-gon), and q is the number of p-gons around a vertex, then the interior angle of each p-gon is $(p - 2)\pi/p$, and around each vertex we have the sum of angles $q(p - 2)\ \pi/p = 2\pi$ to make a circle. This reduces to $(p - 2)(q - 2) = 4$. Thus, for the 3-gon (triangle) we have $q = 6$, for the 4-gon $q = 4$, for the 6-gon $q = 3$, and no other regular polygon covers a surface by itself.

However, this is not the case in the hyperbolic plane. Here, the sum of the angles of a triangle is always less than π. Following the same analysis, this makes $(p - 2)(q - 2) > 4$. So we are able to cover a hyperbolic plane by other regular p-gons such as pentagons (figure 7). Here, we have four regular pentagons fitting exactly around each vertex in this non-Euclidean space.

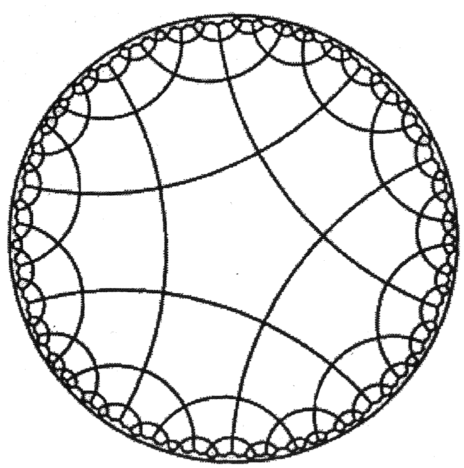

Figure 7. Tiling of a hyperbolic plane by regular pentagons.

The relationships between Euclidean and hyperbolic geometries can be seen by comparing two works by M. C. Escher, both showing alternating angels and devils (figures 8 and 9) [10]. The Euclidean pattern is just a piece of the whole, which could be extended infinitely. In contrast, the non-Euclidean pattern presents the entire world of angels and devils, bounded by a circle.

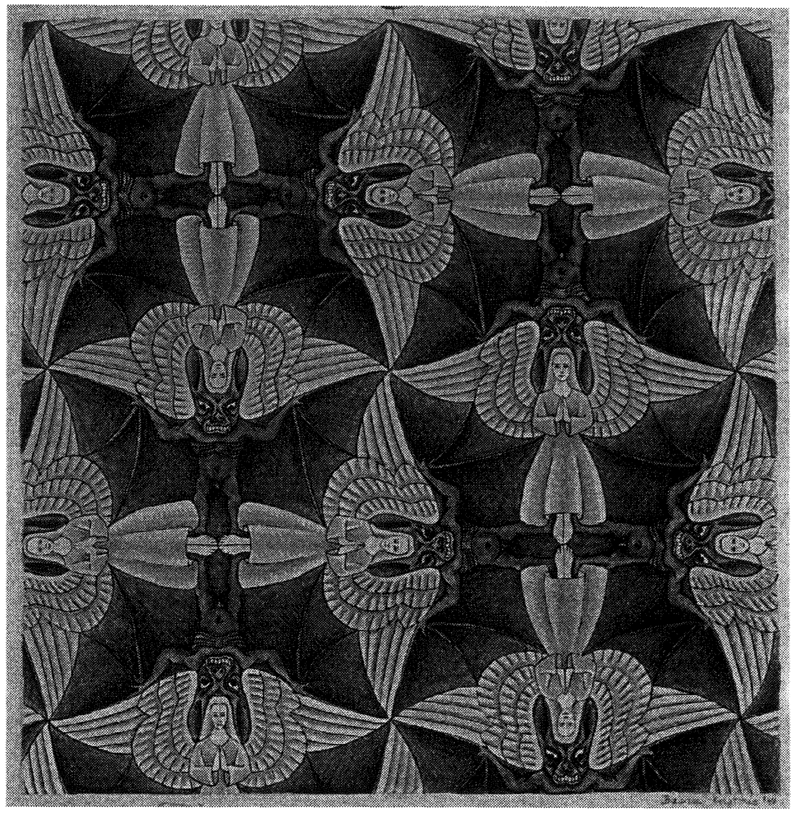

Figure 8.
An *Angel-Devil* drawing on the Euclidean plane.
M. C. Escher's "Symmetry Drawing E45" ©1998 Cordon Art B.V.-Baarn-Holland. All rights reserved.

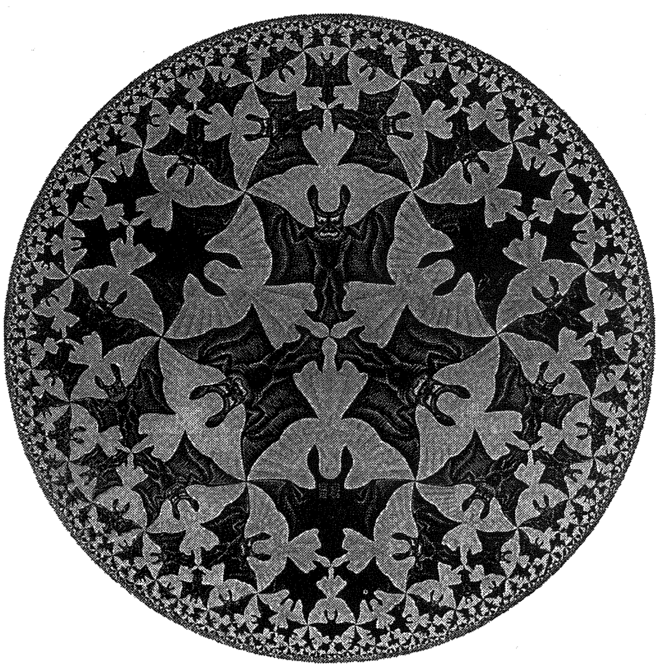

Figure 9.
A non-Euclidean *Angel-Devil* pattern, which fills the unit circle to infinity, covering the entire hyperbolic plane.
M. C. Escher's "Circle Limit IV" ©1998 Cordon Art B.V.-Baarn-Holland. All right reserved.

Earth and Heaven

In Aristotle's *Metaphysics*, Eudoxus's astronomy is described with circles and spheres [11].

> Eudoxus supposed that the motion of the sun or of the moon involves, in either case, three spheres, of which the first is the sphere of the fixed stars, and the second moves in the circle which runs along the middle of the zodiac, and the third in the circle which is inclined across the breadth of the zodiac; but the circle in which the moon moves is inclined at a greater angle than that in which the sun moves. And the motion of the planets involves, in each case, four spheres, and of these also the first and second are the same as the first two mentioned above (for the sphere of the fixed stars is that which moves all the other spheres, and that which is placed beneath this and has its movement in the circle which bisects the zodiac is common to all), but the poles of the third sphere of each planet are in the circle which bisects the zodiac, and the motion of the fourth sphere is in the circle which is inclined at an angle to the equator of the third sphere; and the poles of the third sphere are different for each of the other planets, but those of Venus and Mercury are the same.

The geometrical descriptions of the astronomical heavens were paralleled six centuries later by Plotinus, in a very mystical discussion of where souls go [12].

> There is, we may put it, something that is center; about it, a circle of light shed from it; round center and first circle alike, another circle, light from light; outside that again, not another circle of light but one which, lacking light of its own, must borrow. The last we may figure to ourselves as a revolving circle, or rather a sphere, of a nature to receive light from that third realm, its next higher, in proportion to the light which that itself receives.

This view of heaven and earth in terms of circles is evident in the construction of the great Roman temple to the gods and the planets, the Pantheon [13]. Its dome exactly contains a sphere, with an open circle on top to communicate with heaven (and provide light).

Almost in an echo of the eastern philosophy of the wheel of fate and life, Shakespeare uses the circle to indicate the recurrence of circumstances, or fate. The bastard Edmund recognizes that his amoral victories have only brought him back down to nothing in the final scene of *King Lear* [14].

Edgar: My name is Edgar and thy father's son.
The gods are just, and of our pleasant vices
Make instruments to scourge us.
The dark and vicious place where thee he got
Cost him his eyes.
Edmund: Th' hast spoken right; 'tis true.
The wheel is come full circle; I am here.

Virtues associated with the circle include the thirteenth-century construction in Winchester, England, of *King Arthur's Round Table*, now mounted on the wall of the Great Hall. This was taken as a symbol both of the sacred and of knightly equality under God [15].

The design and appearance are circular for both the main part and the tower of St. Sepulchre's church, in Cambridge, England (c. 1130) [16].

The circle is also the basis for the intricate "rose" window patterns, such as that seen on the south front of the gothic Cathedral of Chartres, France [17].

The *New Jerusalem* pattern is a circle surrounded by twelve smaller ones (in the ratio of the earth to the moon), arranged in a pattern like the numbers on a clock face (figure 10). The twelve circles are grouped in threes, with a dodecahedron circumscribed tangent to them [15]. The number twelve has been seen as a desirable divisor for circles since Babylonian times. It is worth noting that this is exactly double the number which is mathematically required for the symmetric version of this goal, described below.

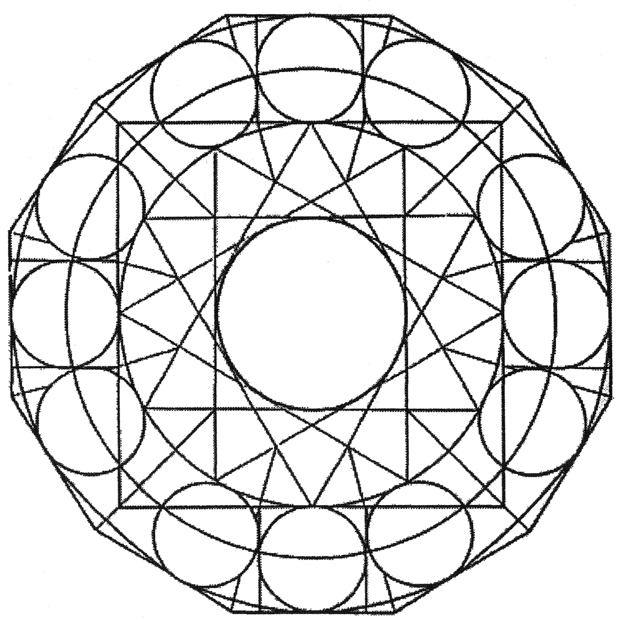

Figure 10. The *New Jerusalem* pattern of twelve circles around one.

The Kissing Number

An interesting measure of the size of a circle is the possible number of neighboring circles. One can place circles in a rectangular grid, with identical circles touching above, below, to the left, and to the right of the central circle. However, a tighter arrangement can be made, with hexagonal symmetry. Here, each circle is surrounded by six equivalent ones. This arrangement has the special property that all the neighbors are touching two other circles off the original one. In other words, six circles will exactly surround one, in two dimensions. Items forced on to a flat surface, if short on space, will get six neighbors, and tend to take on the shapes of hexagons themselves, as in a honeycomb. Another example in a plane is carbon rings in graphite (in pencil "lead"). Here, layers are made of carbon atoms forming three bonds in the plane to similar atoms. The atoms form flat hexagonal rings, which are surrounded by identical hexagons.

However, a group of 60 carbon atoms will wrap itself into a sphere (changing some hexagons into pentagons). This forms a compound called buckminsterfullerene, or a *buckyball* (figure 11). This is named after the inventor of the similarly shaped geodesic dome, Buckminster Fuller.

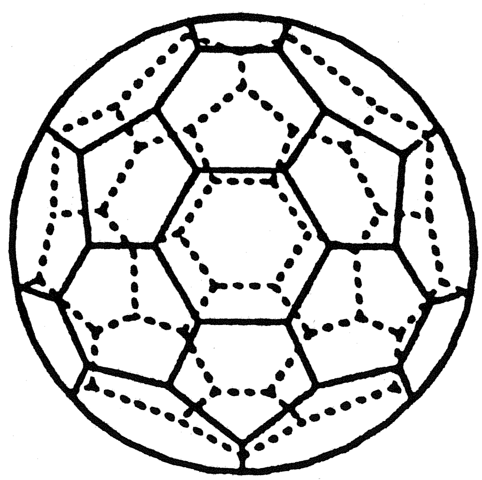

Figure 11. A *buckyball*, or buckminsterfullerene, the spherical C_{60} form of pure carbon atoms, Registry number [*99685-96-8*], in the Chemical Abstracts Service [18].

The problem of stacking or packing balls efficiently has been an interesting puzzle for four centuries. While people have stacked things for millennia, it was in 1611 that Kepler posed the *Sphere-Packing Problem* [19]. What kind of stacking of spheres can be proven to be the densest possible?

A first layer of spheres can be arranged in the rectangular or hexagonal patterns of circles in a plane, described above. Such layers can be stacked exactly atop one another, yielding respectively the arrangements called the body-centered (or 3-dimensional) cubic lattice, and the face-centered cubic lattice or cubic close-packed form. However, if flat layers are stacked repeatedly in a staggered way, a third, most dense pattern emerges, called the hexagonal lattice, or hexagonal close-packed [20].

All three of these packings are found in nature, as are mixtures of these with less regular forms. Interestingly, elemental metals use all forms, in patterns that do not always correlate with electronic symmetry. For example, potassium, chromium and tungsten prefer the body-centered cubic form, as does iron at room temperature. However, iron at other temperatures takes on the cubic close-packed form, as do copper, silver, and gold. In contrast, the elements in the same period directly below iron (i.e., ruthenium and osmium) prefer the hexagonal close-packed arrangement, as do most of the rare-earth elements, as well as zinc and titanium [20].

The number of circles that can surround a circle in any given dimension is called the "kissing number" by mathematicians. On a one-dimensional line, this is two (left and right). In a two-dimensional plane, as we discussed, the kissing number is six. The kissing number is 12 in three dimensions, as in the above hexagonal close-packed arrangement (with six in the plane, three more above, and three others below). In theory, a four-dimensional sphere should have a kissing number of 24, in eight dimensions it is 240, and in 24-dimensional space the kissing number is 196,560 circles touching the center circle at once [19].

The kissing number of six for a circle means that it is possible to construct a regular hexagon (and consequently a six-pointed star) by using the radius to cut the circle into six equal arcs. Unlike the simplicity of the construction of a six-pointed star, it is a challenge to use only a compass and straight edge to construct a pentagram, a five-pointed star extending from a regular pentagon.

To do this task, we need to know how to find the "Golden Mean" of a line segment. The Golden Mean is a cut in a line segment such that the ratio of the larger section to the smaller one is the same as the ratio of the entire line segment to the larger one. It is possible to find the Golden Mean of a given line segment using a compass and straight edge. The Golden Mean has been used extensively in art and architecture from the ancient times of the Egyptians and Greeks up to today. The larger section of the Golden Mean of the radius of any circle divides the circle into ten equal arcs, which enables us to construct the pentagram [21].

The complexity of these procedures, or the harmony and order of the completed form, created the association of mysticism with the combination of a pentagram inscribed in a circle, the chosen symbol of the order of the Pythagoreans, sometimes associated in western tradition with witchcraft (figure 12).

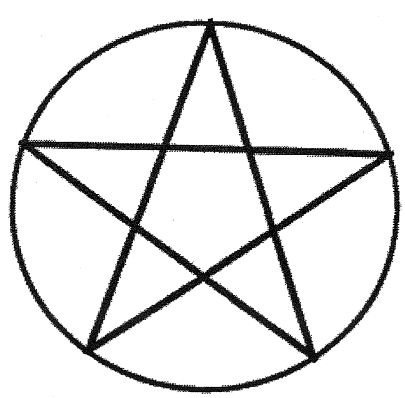

Figure 12. The pentagram of the Pythagoreans.

Poetry of the Whirling Dervishes

The word *Sama* means the joyous but ceremonial religious excitement of the Sufi faithful, either individually or together. Today, it also is any of the famous whirling and circling dances done to the lamenting reed pipe and the pacing drum, performed by the Mevlevi dervishes. These are the disciples of the Sufi poet and spiritual leader, Jalal al-Din Molavi al-Rumi (1207-1273). Rumi is, by one account, the best-selling poet in the United States today [22].

Rumi is one of the most well-known of the historical Persian-language poets, and his mystical poems have been sung from Afghanistan and India to Iran and Turkey for seven centuries. He feels a transcendent, romantic relationship and obsession with the creator, and equally with all of creation. He was born in Balkh (now in Afghanistan). The Mongol invasions caused his family to flee, while he was still a child, to Konya (now part of Turkey), where he lived and died.

Rumi was a theologian, preacher and conventional religious scholar through age 37. Then a man in his sixties arrived, a wandering dervish of wild demeanor named Shams al-Din, who transformed Rumi's world view, despite resistance from Rumi's many conventional religious pupils. Rumi's new ecstasy with the world led him to express his feelings with poetry and with a revision and popularization of the whirling dances of the dervishes. He would even dance and sing to the sounds of the metalworkers in the bazaar.

The *Sama*, in the mystical culture of Persia popularized by Rumi, is demonstrated by a person under the influence of a deep, internal feeling of excitement and love for the spiritual beauty of creation. The person stands up, without feeling self-conscious, and starts circling and whirling. This is especially done according to certain conventions of hand positions and other dance movements that symbolize the spiritual attitude of the obsession with love of the universe. To this day, this is performed in Turkey every year for thousands of visitors, during the week around December 17, the anniversary of Rumi's death.

The Mevlevi dervishes consider the space in the house of the *Sama* to represent a spherical form of the universe. An imaginary axis divides it into western and eastern pieces. At the center is a small circle, called the "pole." After music and singing to candle-light, the circle dancing is done as shown (figure 13). The hands are open and arms extended. The right palm faces up to the spiritual sources of heaven, while the left palm faces down to the world, with the dervish's heart in between, as a medium or bridge. The left foot is fixed, while the right foot causes the body to circle.

Rumi's poetry expresses circles in several different ways, such as in how he saw himself in the world [23].

> Bring into motion your amber-scattering trees;
> bring into dancing the souls of the Sufis.
> Sun, moon, and stars dancing around the circle,
> we dancing in the midst — set that midst a-dancing.
> Your grace minstrelwise with the smallest melody
> brings into the wheel the Sufi of heaven.

Other poems show how circles add to the mysticism of his poetry [24].

Figure 13. The religious devotional activity of the "whirling dervish" disciples of Rumi.

> The wheel of heaven, with all its pomp and splendor,
> circles around God like a mill.
> My soul, circumambulate around such a *Kaaba*;
> beggar, circle about such a table.

Rumi also sees another circle, the circle of life [25].

> Flow down and down in always
> widening rings of being.

To Rumi, we are a drop in the sea, at the moment wandering alone, but eventually destined to return to the sea of absolute truth and love. We were a part of that absolute, then are part of the soil as a plant, animal, or human, then we return to the absolute [25].

There's a strange frenzy in my head,
of birds flying,
each particle circulating on its own.
Is the one I love *everywhere*?

The great and real change in a person is within, not externally visible [25].

You have said what you are;
I am what I am.
Your actions in my head,
my head here in my hands
with something circling inside.
I have no name
for what circles
so perfectly.

Dancing in a circle is common in the cultures of Native Americans, Europeans, and others also. But rarely is it accompanied by poetry that expresses so fully the symbolic meaning of the circling.

Magic with Circles

Not all references to the circle are benign. Dante Alighieri, in the *Inferno*, from his *La Divina Commedia*, described the circles of Hell. For two centuries, the dark side of the search for knowledge has been associated with Goethe's *Faust* [26]. In one scene, a witch draws a circle to cast a spell:

Mephistopheles [to Faust]: My friend, learn well and understand,
This is the way to take a witch in hand.
The Witch: Now, gentlemen, what say you I shall do?
Mephistopheles: A good glass of the well-known juice,
Yet I must beg the oldest sort of you.
A double strength do years produce.
The Witch: With pleasure! Here I have a bottle
From which I sometimes wet my throttle,
Which has no more the slightest stink;
I'll gladly give a little glass to you.

[In a low tone.]
And yet this man, if unprepared he drink,
He can not live an hour, as you know too.

Mephistopheles: He is a friend of mine whom it will profit well;
I would bestow your kitchen's best on him.
So draw your circle, speak your spell,
Give him a cup full to the brim!

[The Witch with curious gestures draws a circle and places marvelous things in it; meanwhile the glasses begin to ring, the cauldron to sound and make music. Lastly, she brings a large book and places the Apes in a circle so as to make them serve as reading-desk and hold the torch. She beckons Faust to come near her.]

Faust [to Mephistopheles]: What is to come of all this? Say!
These frantic gestures and this crazy stuff?
This most insipid, fooling play,
I've known and hated it enough.
Mephistopheles: Nonsense! She only wants to joke us;
I beg you, do not be so stern a man!
Physician-like, she has to play some hocus-pocus
So that the juice will do you all the good it can.
[He obliges Faust to step into the circle.]

While European witchcraft has often been associated with circles, the circle is used for good magic in a tale from the Grimm brothers [27].

When the miller got home, his wife said, "Tell me, from whence comes this sudden wealth into our house?" He answered, "It comes from a stranger who promised me great treasure. I, in return, have promised him what stands behind the mill; we can very well give him the big apple-tree for it." "Ah, husband," said the terrified wife, "that must have been the devil! He did not mean the apple-tree, but our daughter, who was standing behind the mill sweeping the yard."

The miller's daughter was a beautiful, pious girl, and lived through the three years in the fear of God and without sin. When therefore the time was over, and the day came when the Evil-one was to fetch her, she washed herself clean, and made a circle round herself with chalk. The devil appeared quite early, but he could not come near to her.

Circling Back to Idealism

Just as did the ancient Greeks, the philosophers of the seventeenth century used the circle as an example of mathematical idealization, to distinguish between an idea and an actual item. Consider that, in 1690, Locke wrote the following [28].

> Hence the reality of mathematical knowledge. . . . The mathematician considers the truth and properties belonging to a rectangle or circle only as they are an idea in his own mind. For it is possible he never found either of them existing mathematically, i.e., precisely true, in his life. But yet the knowledge he has of any truths or properties belonging to a circle, or any other mathematical figure, are nevertheless true and certain, even of real things existing: because real things are no further concerned, nor intended to be meant by any such propositions, than as things really agree to those archetypes in his mind.

Although the mathematically pure circle doesn't exist in the tangible world, it is present in aspects of many parts of life, as Walt Whitman wrote in *Leaves of Grass* [29].

> Facing west from California's shores,
> Inquiring, tireless, seeking what is yet unfound,
> I, a child, very old, over waves, towards the house of maternity,
> the land of migrations, look afar,
> Look off the shores of my Western sea, the circle almost circled;
> For starting westward from Hindustan, from the vales of Kashmere,
> From Asia, from the north, from the God, the sage, and the hero,
> From the south, from the flowery peninsulas and the spice islands,
> Long having wander'd since, round the earth having wander'd,
> Now I face home again, very pleas'd and joyous,
> (But where is what I started for so long ago?
> And why is it yet unfound?)

To us, the poet here is seeing circles in many aspects of life. He is connecting the end and the beginning of his own travels, of his lifetime, of geographical longitudes, and of the spread of human migration and cultural diversity. He is commenting on his own aspirations and those of all people. The goals and activities of life do not always go forward in a straight line, but can circle one back to the start. Whitman's poem interweaves many of the ideas we see connected to circles.

The philosophy of living in harmony with circular patterns was expressed more recently by Black Elk (*Hehaka Sapa*, 1863-1950), a holy man of the Oglala Sioux Native Americans [30].

> Everything an Indian does is in a circle, and that is because the power of the world always works in circles, and everything tries to be round. In the old days when we were a strong and happy people, all our power came to us from the sacred hoop of the nation, and so long as the hoop was unbroken the people flourished.
>
> Even the seasons form a great circle in their changing, and always come back again to where they were. The life of a man is a circle from childhood to childhood and so it is in everything where power moves.

Our teepees were round like the nests of birds, and these were always set in a circle, the nation's hoop.

In this circle, O ye warriors
Lo, I tell you, each his future.
All shall be as I now reveal it
In this circle; Hear Ye!

Song of the Seer

The circle remains an intimate part of human culture, from math and science to art and human views of the world. Edwin Markham (1852-1940) expressed this in his best-loved poem [31].

He drew a circle that shut me out—
Heretic, rebel, a thing to flout.
But Love and I had the wit to win:
We drew a circle that took him in.

References

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[1b] "Mandalas," Grolier Encyclopedia, 1995.

[2a] James Barr, *Journal of the American Academy of Religion*, "The Question of Religious Influence – The Case of Zoroastrianism, Judaism, and Christianity," LIII / 2, June, 1985, 201.

[2b] Farhang Mehr, *The Zoroastrian Tradition, An Introduction to the Ancient Wisdom of Zarathustra*, Element Inc., Rockport, MA, 1991.

[3] *Sacred Architecture*, A. T. Mann, p. 47, Barnes & Noble Books, 1996.

[4] Other Native Americans included the circular form in their constructions. "Mound Builders refers to the peoples who constructed earthen mounds in North America during prehistoric times (especially to about AD 400). . . . The largest and most elaborate mounds, were often associated with complex geometric earthworks in the form of circles, rectangles, octagons, parallel walls, hilltop fort-like constructions, and other forms." Grolier Encyclopedia, 1995.

[5] *Bible*, King James Version, Isaiah 40:18,21-22.

[6] *The Oxford Dictionary of Quotations*, Angela Partington, ed., Oxford University Press, 4th ed., 1992, p. 16, no. 17.

[7] *Mathematics*, David Bergamini, Time, Inc., 1963, p.44.

[8] *Contemporary Abstract Algebra*, Joseph A. Gallian, 4th ed., Houghton Mifflin Co., NY, 1998, p. 458.

[9a] In Poincaré's model, the empty space outside the boundary disk can be considered as a region for centers of circles orthogonal to the bounding circle. Let P be a point outside this

boundary circle, and *PT* be a line passing through *P* and tangent to the bounding circle at *T*. Then the arc of a circle with center *P* and radius *PT* is itself a line in the Poincaré model.

[9b] Kappraff, *ibid.*, p.176.

[10] *Visions of Symmetry: Notebooks, Periodic Drawings, and Related Work of M. C. Escher*, Doris Schattschneider, W. H. Freeman and Co., New York, 1990.

[11] *Metaphysics*, Aristotle, (Book 12, section 8), 350 BC.

[12] *The Six Ænneads, Tractate 3 - Are the Stars Causes?* Plotinus, section 17., 250 AD.

[13] Mann, *ibid.*, p. 43.

[14] *The Tragedy of King Lear*, W. Shakespeare, 1606.

[15a] Mann, *ibid.*, fig. 69-70.

[15b] Kappraff, *ibid.*, pp. 5-6.

[16] Mann, *ibid.*, p. 101.

[17a] Mann, *ibid.*, p. 144-145.

[17b] *Chartres*, Émile Mâle, Harper & Row, 1983.

[17c] *Rose Windows*, Painton Cowen, Thames and Hudson Ltd., London, 1979, p. 126.

[18] *Aldrich Catalog Handbook of Fine Chemicals*, 1996, item 37,964-6, Milwaukee, WI.

[19] *The Problems of Mathematics*, Ian Stewart, 2nd ed., Oxford University Press, 1992, "Sphereful Symmetry," chapter 6.

[20] *Advanced Inorganic Chemistry*, F. A. Cotton and G. Wilkinson, Fourth edition, John Wiley & Sons, New York, 1980, chapter 1.

[21] *The Golden Section*, Garth E. Runion, Dale Seymour Publ., Palo Alto, CA, 1990, p. 15.

[22] The most recent translation of Rumi's poetry (*The Essential Rumi*, by Coleman Barks) has sold more than a quarter of a million copies. *The Christian Science Monitor*, Alexandra Marks, November 25, 1997, page 1.

[23] *Mystical Poems of Rumi, 1, Translated from the Persian*, A. J. Arberry, University of Chicago Press, Chicago, 1968, poem 23.

[24] Arberry, *ibid.*, poem 32. The *Kaaba* is the black rock cube in Mecca, toward which the Islamic pray as a symbol of heaven. This is also the object that a religious person must circle around repeatedly in the symbolic ritual of the hajj pilgrimage.

[25] *The Essential Rumi*, Translated by Coleman Barks, with John Moyne, A. J. Arberry, Reynold Nicholson, Harper San Francisco, 1995.

[26] *Faust*, Johann Wolfgang von Goethe, G. M. Priest, trans., 1808.

[27] "The Girl Without Hands," *Grimm's Household Tales*, Jakob and Wilhelm Grimm, Edgar Taylor, trans., 1812.

[28] *An Essay Concerning Human Understanding*, John Locke, 1690.

[29] "Facing West from California's Shores," *Leaves of Grass*, Walt Whitman, 1855.

[30] *Familiar Quotations*, John Bartlett, Emily Morison Beck, ed., Little Brown and Co., Boston, 1980, "Black Elk Speaks," p. 701.

[31] *Ibid.*, "Outwitted," Edwin Markham, p. 671.