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Mathematical Connections
in Art, Music, and Science

A Symmetry Classification of Columns

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1. Introduction

The Yale art historian George Hersey showed us the columns in Figure 1 and asked us whether the ideas of symmetry breaking could be used to help classify architectural columns. Provoked by this question and the intriguing columns in that figure, we attempted to answer Hersey in the following way. We view a column as a deformed cylinder and column symmetries as the subgroup of the symmetries of the cylinder that preserve the column.

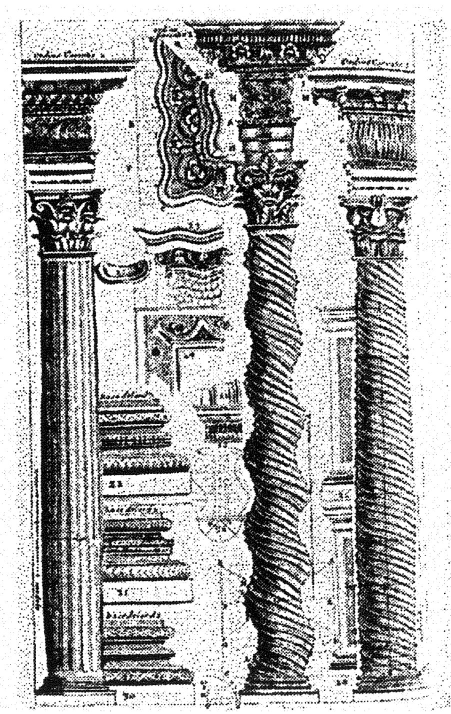


Figure 1: Plate xv from G. Guarini [3]. (Left) Fluted column (see Figure 4). (Right) Spiral column with $k >> 1$ (see Figure 5). (Center) Spiral column with $k = 1$. See Table 2 for a definition of k.

More precisely, we think of a column as a function on a cylinder (either finite or infinite) where the function tells us how far to deform the cylinder in the direction normal to the cylinder. The symmetries of a column are then the symmetries that preserve the level contours of the function, that is, the isotropy subgroup of the defining function. In this paper we present the mathematical classification of the 29 different types of column symmetry. We note that there is a related classification of the *rod groups* that corresponds to the columns with discrete symmetry. See [1].

In a companion paper [2], written with George Hersey, we discuss the question of column symmetry from a historical perspective and attempt to describe the implications of our mathematical classification. The classification theorem is stated and proved in Section 2. Level contours (drawn on a flattened cylinder) of representatives of the twenty-eight nontrivial column symmetry types are presented in Section 3.

2. Symmetries of Columns

We imagine a *column* to be a real-valued function f on the cylinder $\mathcal{C} = \mathbf{S}^1 \times \mathbf{R}$. Let $(\varphi, z) \in \mathcal{C}$. The function $f(\varphi, z)$ measures the height of the column in the direction normal to the cylinder at the point (φ, z).

The group of symmetries of the cylinder is

$$\Gamma = \mathbf{D}_2(\tau, \kappa) \dot{+} (\mathbf{SO(2)} \oplus \mathbf{R})$$

where Γ acts on $(\varphi, z) \in \mathcal{C}$ by

$$\begin{aligned} (\theta, t)(\varphi, z) &= (\varphi + \theta, z + t) \qquad (\theta, t) \in \mathbf{SO(2)} \oplus \mathbf{R} \\ \tau(\varphi, z) &= (-\varphi, z) \\ \kappa(\varphi, z) &= (\varphi, -z). \end{aligned}$$

Multiplication in Γ follows from the definition of the action. Suppose that $(A_j, (\theta_j, t_j))$ is in Γ for $j = 1, 2$, where $A_j \in \mathbf{D}_2$, $\theta_j \in \mathbf{SO(2)}$ and $t_j \in \mathbf{R}$. Then multiplication is given by

$$(A_2, (\theta_2, t_2)) \cdot (A_1, (\theta_1, t_1)) = (A_2 A_1\,,\ A_2(\theta_1, t_1) + (\theta_2, t_2)). \tag{2.1}$$

We wish to classify columns by their symmetries. A *symmetry* of the column $f : \mathcal{C} \to \mathbf{R}$ is $\gamma \in \Gamma$ such that

$$f(\gamma(\varphi, z)) = f(\varphi, z) \quad \forall (\varphi, z) \in \mathcal{C}.$$

The symmetry group $\Sigma_f \subset \Gamma$ is the collection of all symmetries of f. We classify all subgroups Σ which are symmetry subgroups for some column f.

Our classification proceeds as follows. To each subgroup $\Sigma \subset \Gamma$, we can associate the normal subgroup

$$\Sigma_0 = \Sigma \cap (\mathbf{SO(2)} \oplus \mathbf{R}). \tag{2.2}$$

(So Σ_0 consists of the pure 'translations' in Σ.) Thus it suffices to

(i) classify the closed subgroups Σ_0 of $\mathbf{SO(2)} \oplus \mathbf{R}$,

(ii) for each subgroup Σ_0 in (i), compute the subgroups $\Sigma \subset \Gamma$ that satisfy (2.2).

The calculation in (ii) is simplified by observing that Σ is contained in the normalizer of Σ_0.

As usual, we identify conjugate subgroups of Γ. In addition, we identify subgroups that are related by axial scalings. More precisely, we define the scaling transformation $s_\alpha : \Gamma \to \Gamma$ by

$$s_\alpha(A,\theta,t) = (A,\theta,\alpha t), \qquad (A,\theta,t) \in \Gamma.$$

Provided $\alpha \neq 0$, this is an isomorphism. We say that two subgroups Σ, Σ' are related by a scaling if $s_\alpha\Sigma = \Sigma'$ for some nonzero α.

2.1. Classification of Subgroups of $\mathbf{SO(2)} \oplus \mathbf{R}$.

In this section, we classify the closed subgroups of $\mathbf{SO(2)} \oplus \mathbf{R}$ up to scaling and conjugacy in Γ. Also, we compute the normalizers of these subgroups in Γ.

Lemma 2.1 *Suppose that C is a compact subgroup of $\mathbf{SO(2)} \oplus \mathbf{R}$. Then $C \subset \mathbf{SO(2)} \oplus \mathbf{1}$.*

Proof: If $(\theta,t) \in \mathbf{SO(2)}\oplus\mathbf{R}$ and $t \neq 0$, then (θ,t) generates a noncompact subgroup of $\mathbf{SO(2)}\oplus\mathbf{R}$ (isomorphic to $\mathbf{Z}$). It follows that $(\theta,t) \notin C$. ∎

Proposition 2.2 *Suppose that G is a closed connected subgroup of $\mathbf{SO(2)} \oplus \mathbf{R}$. Then, up to conjugacy and scaling, G is one of the subgroups*

$$\mathbf{SO(2)} \oplus \mathbf{R}, \quad \mathbf{SO(2)} \oplus \mathbf{1}, \quad \mathbf{1} \oplus \mathbf{R}, \quad \mathbf{L}, \quad \mathbf{1},$$

where

$$\mathbf{L} = \{(t,t) \in \mathbf{SO(2)} \oplus \mathbf{R} : t \in \mathbf{R}\}.$$

Proof: If $\dim G = 2$, then connectivity implies that $G = \mathbf{SO(2)} \oplus \mathbf{R}$. If $\dim G = 1$, then connectivity implies that G is group isomorphic to either $\mathbf{SO(2)}$ or $\mathbf{R}$. In the first case, it follows from Lemma 2.1 that $G = \mathbf{SO(2)} \oplus \mathbf{1}$. In the second case, there is a smooth isomorphism $h : \mathbf{R} \to G \subset \mathbf{SO(2)} \oplus \mathbf{R}$. This isomorphism is given by $h(t) = (\theta_0 t, a_0 t)$ for some $(\theta_0, a_0) \in \mathbf{SO(2)} \oplus \mathbf{R}$ (defined as $h(1)$). By assumption $a_0 \neq 0$. If $\theta_0 = 0$, then $G = \mathbf{1} \oplus \mathbf{R}$. If $\theta_0 \neq 0$, then by axial scaling we can arrange that $a_0 = \theta_0$ and $G = \mathbf{L}$. ∎

From now on, we use the abbreviations $\mathbf{R} = \mathbf{1} \oplus \mathbf{R}$ and $\mathbf{SO(2)} = \mathbf{SO(2)} \oplus \mathbf{1}$. The proper closed subgroups of $\mathbf{SO(2)}$ are given by $\mathbf{Z}_k$, $k \geq 1$: the subgroup of rotations of the cylinder through angles which are multiples of $2\pi/k$. In addition, we set $\mathbf{Z} \subset \mathbf{R}$ to be the subgroup of unit axial translations of the cylinder generated by the element $(0,1) \in \mathbf{SO(2)} \oplus \mathbf{R}$. Finally, for any $\omega \in \mathbf{R}$, we define

$$\mathbf{N}_\omega = \{(\omega n, n) \in \mathbf{SO(2)} \oplus \mathbf{R} : n \in \mathbf{Z}\}.$$

Of course, $\mathbf{N}_0 = \mathbf{Z}$.

Theorem 2.3 *Up to axial scaling and conjugacy, the closed subgroups $\Sigma_0 \subset \mathbf{SO(2)} \oplus \mathbf{R}$ are listed in Table 1.*

$\dim \Sigma_0$	Σ_0		H
2	$\mathbf{SO(2)} \oplus \mathbf{R}$		$\mathbf{D}_2$
1	$\mathbf{SO(2)}$		$\mathbf{D}_2$
	$\mathbf{SO(2)} \oplus \mathbf{Z}$		$\mathbf{D}_2$
	$\mathbf{Z}_k \oplus \mathbf{R}$		$\mathbf{D}_2$
	$\mathbf{Z}_k \oplus \mathbf{L}$		$\mathbf{Z}_2(\tau\kappa)$
0	$\mathbf{Z}_k$		$\mathbf{D}_2$
	$\mathbf{Z}_k \oplus \mathbf{Z}$		$\mathbf{D}_2$
	$\mathbf{Z}_k \oplus \mathbf{N}_\omega$	$0 < \omega < \pi/k$	$\mathbf{Z}_2(\tau\kappa)$
	$\mathbf{Z}_k \oplus \mathbf{N}_{\pi/k}$		$\mathbf{D}_2$

Table 1: Classification of closed subgroups $\Sigma_0 \subset \Gamma$ up to scaling and conjugacy. The normalizers are given by $N(\Sigma_0) = H \oplus (\mathbf{SO(2)} \oplus \mathbf{R})$

Proof: Since Σ_0 is abelian, we can write $\Sigma_0 \cong C \oplus \mathbf{Z}^p \oplus \mathbf{R}^q$ where C is compact and $p, q \geq 0$. Clearly, $p + q \leq 1$. By Lemma 2.1, $C = \mathbf{SO(2)}$ or $C = \mathbf{Z}_k$.

Assume that $C = \mathbf{SO(2)}$. Since $\mathbf{SO(2)} \oplus \mathbf{R}$ is connected, the only subgroup satisfying $\dim \Sigma_0 = 2$ is $\Sigma_0 = \mathbf{SO(2)} \oplus \mathbf{R}$. Suppose next that $\dim \Sigma_0 = 1$. We claim that $\Sigma_0 = \mathbf{SO(2)}$ or $\Sigma_0 = \mathbf{SO(2)} \oplus \mathbf{Z}$. Choose the smallest positive $t \in \mathbf{R}$ such that there is $\theta \in \mathbf{SO(2)}$ with $(\theta, t) \in \Sigma_0$. Since $(\theta, 0) \subset \Sigma_0$, it follows that $\Sigma_0 = \mathbf{SO(2)} \oplus t\mathbf{Z}$, where $t\mathbf{Z}$ is the subgroup of $\mathbf{SO(2)} \oplus \mathbf{R}$ generated by $(0, t)$. By making an axial scaling, we can set $t = 1$ so that $\Sigma_0 = \mathbf{SO(2)} \oplus \mathbf{Z}$.

Now assume that $C = \mathbf{Z}_k$. If $\dim \Sigma_0 = 1$, then it follows from Proposition 2.2 that $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{R}$ or $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{L}$. If $\dim \Sigma_0 = 0$, then either $\Sigma_0 = \mathbf{Z}_k$ or $\Sigma_0 \cong \mathbf{Z}_k \oplus \mathbf{Z}$. In the latter case, we can choose a generator $(a, b) \in \mathbf{Z} \subset \mathbf{SO(2)} \oplus \mathbf{R}$ with smallest $b > 0$. Making an axial scaling, we can suppose that the generator is of the form $(\omega, 1)$ for some $\omega \in \mathbf{R}$. In other words, $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{N}_\omega$. Note that $\mathbf{Z}_k \oplus \mathbf{N}_{\omega + 2\pi/k} = \mathbf{Z}_k \oplus \mathbf{N}_\omega$, so we can suppose that $|\omega| \leq \pi/k$. Using formula (2.1) we compute that

$$\tau \cdot (\omega t, t) \cdot \tau^{-1} = (-\omega t, t),$$

where $\tau \cdot (\omega t, t)$ is an abbreviation for $(\tau, (0, 0)) \cdot (1, (\omega t, t))$. Hence up to conjugacy, we may suppose that $\omega \geq 0$. The case $\omega = 0$ is the distinguished case $\mathbf{N}_0 = \mathbf{Z}$. ∎

Proposition 2.4 *The normalizers of the subgroups* $\Sigma_0 \subset \mathbf{SO(2)} \oplus \mathbf{R}$ *have the form*

$$N(\Sigma_0) = H \dot{+} (\mathbf{SO(2)} \oplus \mathbf{R}),$$

where the subgroup $H \subset \mathbf{D}_2$ *is as given in Table 1.*

Proof: Since $\mathbf{SO(2)} \oplus \mathbf{R}$ is abelian, it is clear that $\mathbf{SO(2)} \oplus \mathbf{R} \subset N(\Sigma_0)$. Hence $N(\Sigma_0) = H \dot{+} (\mathbf{SO(2)} \oplus \mathbf{R})$ for some subgroup $H \subset \mathbf{D}_2$. We compute that $A \cdot (\theta, t) \cdot A^{-1}$ is the element $A(\theta, t) \in \mathbf{SO(2)} \oplus \mathbf{R}$. Hence, H consists of those elements $A \in \mathbf{D}_2$ that preserve Σ_0. The element

Σ_0	Σ		
$\mathbf{SO(2)} \oplus \mathbf{R}$	Γ		
$\mathbf{SO(2)}$	$\mathbf{Z}_2(\tau)\dot{+}\mathbf{SO(2)}$	$\mathbf{D}_2\dot{+}\mathbf{SO(2)}$	
$\mathbf{SO(2)} \oplus \mathbf{Z}$	$\mathbf{Z}_2(\tau)\dot{+}(\mathbf{SO(2)} \oplus \mathbf{Z})$	$\mathbf{D}_2\dot{+}(\mathbf{SO(2)} \oplus \mathbf{Z})$	
$\mathbf{Z}_k \oplus \mathbf{R}$	$\mathbf{Z}_2(\kappa)\dot{+}(\mathbf{Z}_k \oplus \mathbf{R})$	$\mathbf{D}_2\dot{+}(\mathbf{Z}_k \oplus \mathbf{R})$	
$\mathbf{Z}_k \oplus \mathbf{L}$	$\mathbf{Z}_k \oplus \mathbf{L}$	$\mathbf{Z}_2(\tau\kappa)\dot{+}(\mathbf{Z}_k \oplus \mathbf{L})$	
$\mathbf{Z}_k$	$\mathbf{Z}_k$	$\mathbf{Z}_2(\tau)\dot{+}\mathbf{Z}_k$	$\mathbf{Z}_2(\kappa) \oplus \mathbf{Z}_k$
	$\mathbf{Z}_2(\tau\kappa)\dot{+}\mathbf{Z}_k$	$\mathbf{D}_2\dot{+}\mathbf{Z}_k$	
$\mathbf{Z}_k \oplus \mathbf{N}_\omega \quad 0 \leq \omega \leq \pi/k$	$\mathbf{Z}_k \oplus \mathbf{N}_\omega$	$\mathbf{Z}_2(\tau\kappa)\dot{+}(\mathbf{Z}_k \oplus \mathbf{N}_\omega)$	
$\mathbf{Z}_k \oplus \mathbf{Z}$	$\mathbf{Z}_2(\tau)\dot{+}(\mathbf{Z}_k \oplus \mathbf{Z})$	$\mathbf{Z}_2(\kappa)\dot{+}(\mathbf{Z}_k \oplus \mathbf{Z})$	$\mathbf{D}_2\dot{+}(\mathbf{Z}_k \oplus \mathbf{Z})$
$\mathbf{Z}_k \oplus \mathbf{N}_{\pi/k}$	$\mathbf{Z}_2(\tau)\dot{+}(\mathbf{Z}_k \oplus \mathbf{N}_{\pi/k})$	$\mathbf{Z}_2(\kappa)\dot{+}(\mathbf{Z}_k \oplus \mathbf{N}_{\pi/k})$	$\mathbf{D}_2\dot{+}(\mathbf{Z}_k \oplus \mathbf{N}_{\pi/k})$

Table 2: The 22 untwisted symmetry groups $\Sigma \subset \Gamma$

$\tau\kappa$ acts as $-I$ on $\mathbf{SO(2)} \oplus \mathbf{R}$ and so is always contained in H. It follows that $H = \mathbf{Z}_2(\tau\kappa)$ or $H = \mathbf{D}_2$. It now suffices to determine whether or not τ preserves Σ_0, that is, whether or not Σ_0 is preserved by the transformation $(\theta, t) \mapsto (-\theta, t)$. ∎

2.2. Untwisted Symmetry Groups

Suppose that $\Sigma \subset \Gamma$ is a symmetry group. Then $\Sigma_0 = \Sigma \cap (\mathbf{SO(2)} \oplus \mathbf{R})$ is one of the subgroups listed in Table 1. We say that Σ is an *untwisted* subgroup of Γ if Σ is conjugate to a subgroup of the form $K\dot{+}\Sigma_0$ where K is contained in the subgroup H given in Table 1. The untwisted symmetry groups are listed in Table 2.

It is not the case that every subgroup $K \subset H$ produces a symmetry group. For example, when $\Sigma_0 = \mathbf{SO(2)} \oplus \mathbf{R}$, the only symmetry group Σ corresponding to Σ_0 is $\Sigma = \Gamma$. (This is independent of the restriction to untwisted symmetry groups.) To verify this point, observe that $\mathbf{SO(2)} \oplus \mathbf{R}$ acts transitively on the cylinder $\mathcal{C}$. Hence if Σ is the symmetry group of a function $f : \mathcal{C} \to \mathbf{R}$, then f is the constant function. It follows that f is invariant under Γ, and that the symmetry subgroup $\Sigma = \Gamma$.

When Σ_0 contains $\mathbf{SO(2)}$, the function f is constant on each horizontal cross-section of $\mathcal{C}$ and hence automatically has the symmetry τ. In these cases, the only possibilities are $K = \mathbf{Z}_2(\tau)$ and $K = \mathbf{D}_2$. Similarly, when Σ contains $\mathbf{R}$ then automatically $\kappa \in \Sigma$ and the only possibilities are $K = \mathbf{Z}_2(\kappa)$ and $K = \mathbf{D}_2$.

In all other cases, there are no restrictions on K other than the condition $K \subset H$.

2.3. Twisted Symmetry Groups

We continue to suppose that Σ is a symmetry group with $\Sigma_0 = \Sigma \cap (\mathbf{SO(2)} \oplus \mathbf{R})$. We have $\Sigma \subset H\dot{+}(\mathbf{SO(2)} \oplus \mathbf{R})$ where H is given in Table 1. The canonical projection $\pi : \Gamma \to D_2$ induces a projection $\pi : \Sigma \to H$.

We say that a symmetry group $\Sigma \subset \Gamma$ is *twisted* if it is not conjugate to an untwisted symmetry group. Equivalently, there exists an $A \in \pi(\Sigma)$ such that $A \notin \Sigma$.

The next lemma states that, without loss of generality, we can always suppose that the element $A = \tau\kappa$ is not responsible for twisting.

Lemma 2.5 *Suppose that Σ is a symmetry group and that $\tau\kappa \in \pi(\Sigma)$. Then there is a subgroup of Γ that is conjugate to Σ and contains $\tau\kappa$. The conjugacy leaves Σ_0 unchanged.*

Proof: Recall that $\tau\kappa$ acts as $-I$ on $\mathbf{SO(2)} \oplus \mathbf{R}$. By assumption $(\tau\kappa, \theta, t) \in \Sigma$ for some $(\theta, t) \in \mathbf{SO(2)} \oplus \mathbf{R}$. We conjugate by the element $(-\theta/2, -t/2) \in \mathbf{SO(2)} \oplus \mathbf{R}$. Compute that

$$(1\,,\,(-\theta/2, -t/2)) \cdot (\tau\kappa\,,\,(\theta, t)) \cdot (1\,,\,(\theta/2, t/2)) = (\tau\kappa\,,\,(0,0)),$$

as required. ■

Proposition 2.6 *Let Σ be a twisted symmetry group. Then either $\Sigma_0 = \mathbf{Z}_k$, $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{Z}$ or $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{N}_{\pi/k}$. In addition, $\pi(\Sigma)$ is one of the three subgroups $\mathbf{Z}_2(\tau)$, $\mathbf{Z}_2(\kappa)$ and $\mathbf{D}_2$.*

Remark: The possibility $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{N}_{\pi/k}$ will be eliminated in the proof of Theorem 2.7.

Proof: It follows from Lemma 2.5 that we can eliminate the subgroups Σ_0 for which $H = \mathbf{Z}_2(\tau\kappa)$, that is we can eliminate $\mathbf{Z}_k \oplus \mathbf{L}$ and $\mathbf{Z}_k \oplus \mathbf{N}_\omega$.

Next, suppose that Σ_0 contains $\mathbf{SO(2)}$. As observed in the previous subsection, Σ contains τ. If Σ is larger than $\mathbf{Z}_2(\tau)\dot{+}\mathbf{SO(2)}$, then $\pi(\Sigma) = \mathbf{D}_2$. It follows from Lemma 2.5 that $\tau\kappa \in \Sigma$ and hence $\Sigma = \mathbf{D}_2\dot{+}\Sigma_0$. In either case, Σ is untwisted. The possibility that Σ_0 contains $\mathbf{R}$ can be eliminated similarly. This completes the proof that Σ_0 is one of the groups $\mathbf{Z}_k$, $\mathbf{Z}_k \oplus \mathbf{Z}$ or $\mathbf{Z}_k \oplus \mathbf{N}_{\pi/k}$.

Recall that $\pi(\Sigma)$ is a subgroup of $\mathbf{D}_2$. If $\pi(\Sigma) = \mathbf{1}$, then $\Sigma = \Sigma_0$. If $\pi(\Sigma) = \mathbf{Z}_2(\tau\kappa)$, then Σ is conjugate to $\mathbf{Z}_2(\tau\kappa)\dot{+}\Sigma_0$ by Lemma 2.5. Hence, for Σ to be twisted, $\pi(\Sigma)$ must be one of the three remaining subgroups of $\mathbf{D}_2$. ■

Theorem 2.7 *Up to conjugacy and scaling, there are seven twisted symmetry groups in Γ. These are as listed in Table 3.*

Proof: By Proposition 2.6, we can assume that $\Sigma_0 = \mathbf{Z}_k$, $\mathbf{Z}_k \oplus \mathbf{Z}$ or $\mathbf{Z}_k \oplus \mathbf{N}_{\pi/k}$ and that $K = \pi(\Sigma)$ is one of the subgroups $\mathbf{Z}_2(\tau)$, $\mathbf{Z}_2(\kappa)$ or $\mathbf{D}_2$. We consider the three possibilities for K in turn.

Suppose that $K = \mathbf{Z}_2(\tau)$. Then $\sigma = (\tau, (\theta, t)) \in \Sigma$ for some $(\theta, t) \in \mathbf{SO(2)} \oplus \mathbf{R}$. Conjugating by $(-\theta/2, 0) \in \mathbf{SO(2)} \oplus \mathbf{R}$, we can set $\theta = 0$. Note that

$$\sigma^2 = (1, (0, 2t)) \in \Sigma_0.$$

When $\Sigma_0 = \mathbf{Z}_k$, it follows that $t = 0$ in which case $\sigma = \tau$, and there is no twisting. When $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{Z}$, there is the additional possibility that $2t \in \mathbf{Z}$ but $t \notin \mathbf{Z}$. Since $\mathbf{Z} \subset \Sigma$, this reduces to the case $t = 1/2$. The argument is more complicated when $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{N}_{\pi/k}$. Squaring yields the condition $(0, 2t) \in \mathbf{Z}_k \oplus \mathbf{N}_{\pi/k}$. Working modulo $\mathbf{Z}_k \oplus \mathbf{N}_{\pi/k}$, we can choose σ so that $t = 1$. But

Σ_0	$\pi(\Sigma)$	generators of Σ/Σ_0
$\mathbf{Z}_k$	$\mathbf{Z}_2(\kappa)$	$\tilde{\kappa}$
	$\mathbf{D}_2$	$\tau,\quad \tilde{\kappa}$
$\mathbf{Z}_k \oplus \mathbf{Z}$	$\mathbf{Z}_2(\tau)$	$\tilde{\tau}$
	$\mathbf{Z}_2(\kappa)$	$\tilde{\kappa}$
	$\mathbf{D}_2$	$\tau,\quad \tilde{\kappa}$
	$\mathbf{D}_2$	$\tilde{\tau},\quad \kappa$
	$\mathbf{D}_2$	$\tilde{\tau},\quad \tilde{\kappa}$

Table 3: The 7 twisted symmetry groups $\Sigma \subset \Gamma$. Σ is generated by Σ_0 together with the generators of Σ/Σ_0. Notation: $\tilde{\kappa} = (\kappa, (\pi/k, 0))$, $\tilde{\tau} = (\tau, (0, 1/2))$

still working modulo $\mathbf{N}_{\pi/k}$, we can replace σ by $\sigma = (\tau, (\pi/k, 0))$. Conjugating once again, we have $\sigma = \tau$ and there is no twisting.

The case $K = \mathbf{Z}_2(\kappa)$ is similar. Conjugation reduces to $\sigma = (\kappa, (\theta, 0))$ and squaring yields the condition $2\theta \in \mathbf{Z}_k$. Twisting occurs when $\theta = \pi/k$ but only for $\Sigma_0 = \mathbf{Z}_k$ and $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{Z}$.

Finally, suppose that $K = \mathbf{D}_2$. We concentrate attention on the two generators

$$\sigma_1 = (\tau, (\theta_1, t_1)) \qquad \sigma_2 = (\kappa, (\theta_2, t_2))$$

of Σ modulo Σ_0. Since the reflections are orthogonal, we can simultaneously conjugate so that $\theta_1 = t_2 = 0$. Squaring the generators, we obtain that $\theta_2 \in \mathbf{Z}_{2k}$ and either $t_1 = 0$, $2t_1 \in \mathbf{Z}$ or $t_1 \in \mathbf{Z}$ depending on whether $\Sigma_0 = \mathbf{Z}_k$, $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{Z}$ or $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{N}_{\pi/k}$. The various combinations of generators yield one untwisted subgroup and one twisted subgroup for $\Sigma_0 = \mathbf{Z}_k$, and one untwisted subgroup and three twisted subgroups for $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{Z}$. Once again, there is no twisting when $\Sigma_0 = \mathbf{Z}_k \oplus \mathbf{N}_{\pi/k}$. The arguments are similar to the previous cases of K; we replace σ_j by untwisted group elements. ∎

3. Classification of Columns

The results of the previous section show that there are twenty-nine symmetry classes of columns. These symmetry classes can be distiguished by a sequence of questions. The most important question is:

Are the symmetries of the column continuous, discrete and infinite, or finite?

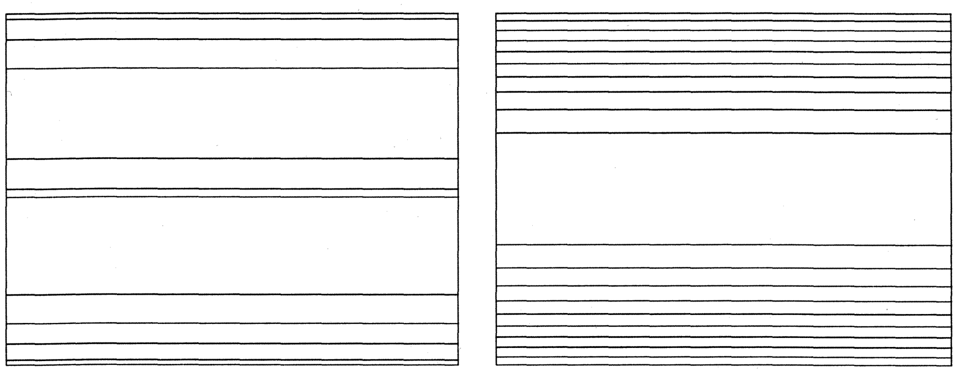
The column has continuous symmetries when the column can be slid along itself. These symmetries can occur either by axial translations, rotations about the axis, or by a combination of the two. With two exceptions infinite discrete symmetry groups occur when the column is axially periodic but has no continuous symmetries. Both of the first two types of symmetry groups are infinite. If the symmetry group of a column is not infinite, then it is finite.

3.1. Columns with Continuous Symmetry

If the column has both axial-translation and rotation symmetry, than the column is a cylinder with symmetry group Γ. Continuous symmetries come in three types: rotations about the column axis (columns of revolution), translations along the column axis (fluted columns), or corkscrew symmetries which are a mixture of the two (spiral columns).

3.1.1. Columns of Revolution — Four Types

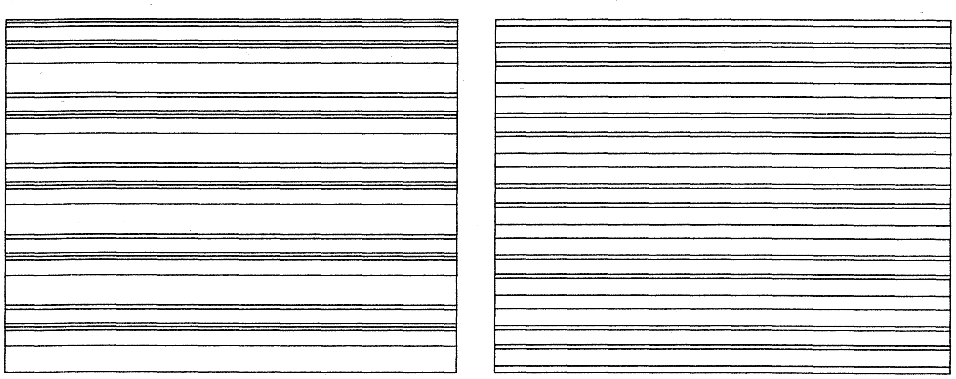
There are four types of column with rotational $\mathbf{SO(2)}$ symmetry. Two types are periodic in the axial direction and two are not. The nonperiodic columns may have a reflection symmetry in the horizontal plane ($\mathbf{D}_2\dot{+}\mathbf{SO(2)}$ — for example a column which is bowed out at the center) or not ($\mathbf{Z}_2(\tau)\dot{+}\mathbf{SO(2)}$ — a column which widens at the base). See Figure 2.



(a) (b)

Figure 2: Nonperiodic columns of revolution. (a) No up-down reflection; (b) Up-down reflection.

The periodic columns of revolution may have an up-down symmetry ($\mathbf{D}_2\dot{+}(\mathbf{SO(2)} \oplus \mathbf{Z})$) or not ($\mathbf{Z}_2(\tau)\dot{+}(\mathbf{SO(2)} \oplus \mathbf{Z})$). See Figure 3.



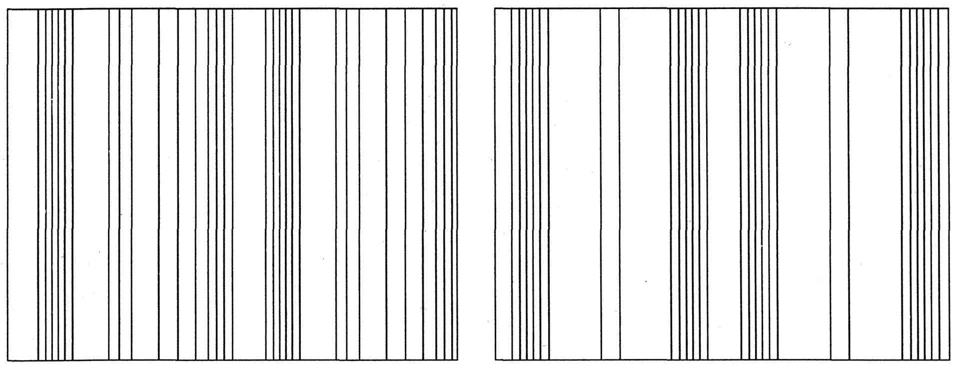
(a) (b)

Figure 3: Periodic columns of revolution. (a) No up-down reflection; (b) Up-down reflection.

3.1.2. Fluted Columns — Two Types

All remaining symmetry groups have at least $\mathbf{Z}_k$ symmetry for some k, that is, rotation symmetry through an angle $2\pi/k$. In our description of this classification we now set $k = 1$ with the understanding that there is a version of each of the remaining columns for each natural number k. Indeed, the pictures we show all have $k = 2$.

There are two types of columns with axial translation symmetry: those which have a plane of reflection across a plane containing the axis of the cylinder ($\mathbf{D}_2\dot{+}\mathbf{R}$) and those that do not ($\mathbf{Z}_2\dot{+}\mathbf{R}$). See Figure 4.

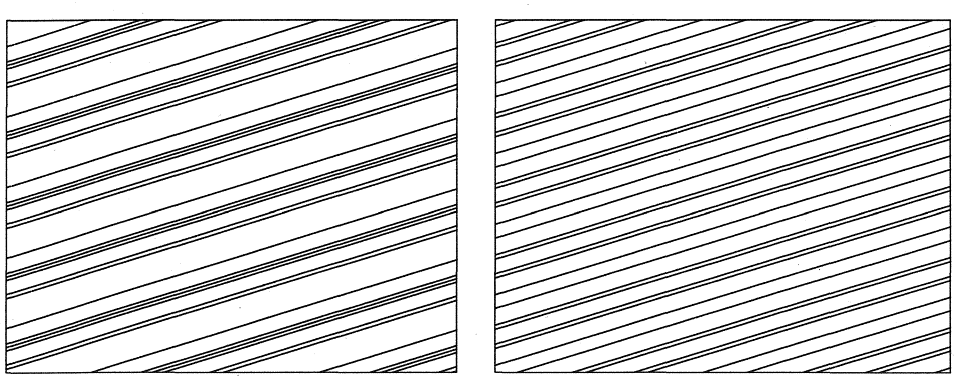


(a) (b)

Figure 4: Fluted columns. (a) No left-right reflection; (b) Left-right reflection.

3.1.3. Spiral Columns — Two Types

There are two types of spirals — both of which have twisted translation symmetry. There are the spirals that are symmetric when the column is rotated by 180° in a plane containing the axis of the cylinder ($\mathbf{Z}_2(\tau\kappa)\dot{+}\mathbf{L}$) and those that do not have this symmetry ($\mathbf{L}$). See Figure 5.



(a) (b)

Figure 5: Spiral columns. (a) No additional symmetry; (b) Up-down rotation.

3.2. Columns with Discrete Symmetry

There are two types of symmetry groups that are infinite and discrete — those with corkscrew symmetries and those without.

3.2.1. Periodic Columns with No Corkscrew Symmetry — Eight Types

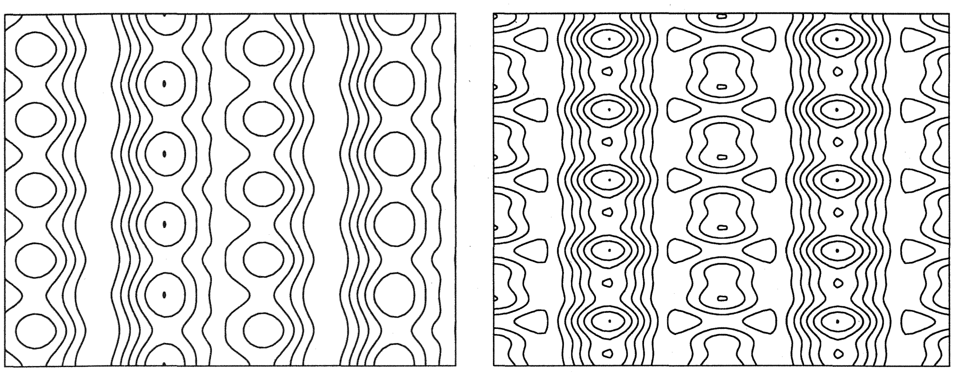
Recall that τ is a reflection through a plane containing the axis of the cylinder and κ is the reflection through the midplane - the up-down symmetry. Each of these symmetries has a glide reflection version

$$\tilde{\tau} = (\tau, (0, 1/2)) \qquad \tilde{\kappa} = (\kappa, (\pi, 0)).$$

There are ten subsets $G \subset \{\tau, \tilde{\tau}, \kappa, \tilde{\kappa}\}$ that form symmetry groups when coupled with $\mathbf{Z}$. These subsets are:

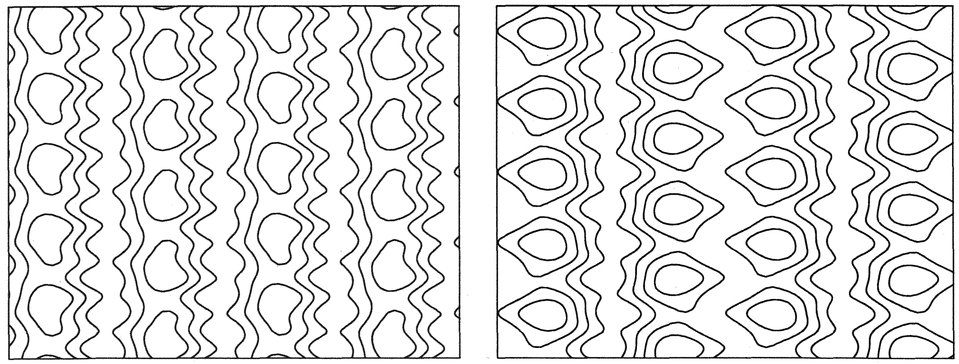
$$\{\kappa\} \quad \{\tau\} \quad \{\tilde{\kappa}\} \quad \{\tilde{\tau}\} \quad \{\tau, \kappa\} \quad \{\tilde{\tau}, \kappa\} \quad \{\tau, \tilde{\kappa}\} \quad \{\tilde{\tau}, \tilde{\kappa}\} \quad \emptyset \quad \{\tau\kappa\}.$$

The symmetry groups of the corresponding periodic columns are: $< G, \mathbf{Z} >$ — the group generated by G and $\mathbf{Z}$. Examples of columns having one pure reflection symmetry are found in Figure 6. Examples of columns having precisely one glide reflection are given in Figure 7. Columns having two reflections or glide reflections are shown in Figure 8. The last two subsets correspond to symmetry groups that lie in infinite families and these infinite families have corkscrew symmetries. See Figure 10.



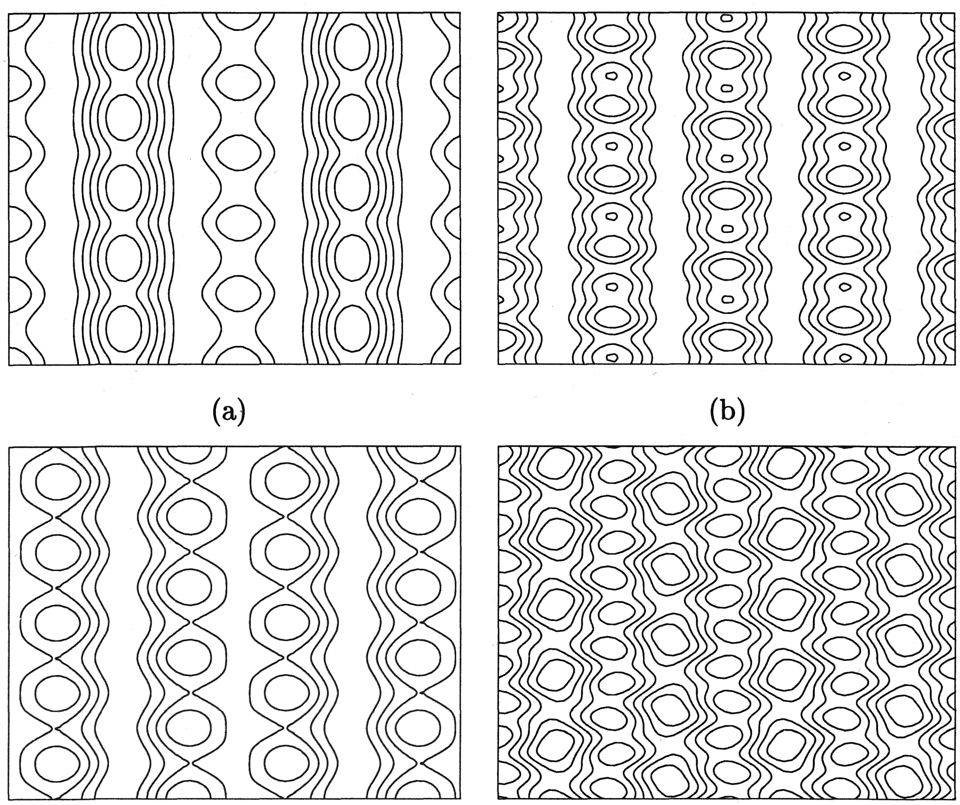
(a) (b)

Figure 6: Periodic columns with one reflection. (a) Up-down symmetric; (b) Left-right symmetric.



(a) (b)

Figure 7: Periodic columns with one glide. (a) Up-down glide; (b) Left-right glide.



(a) (b) (c) (d)

Figure 8: Periodic columns with two reflections or glides. (a) Up-down and left-right reflections; (b) Up-down glide and left-right reflection; (c) Up-down reflection and left-right glide; (d) Up-down glide and left-right glide.

3.2.2. Discrete Corkscrew Columns — Five Types

There are three column types having $\mathbf{N}_\pi$ symmetry. These columns remain the same when translated in the axial direction a unit distance and simultaneously rotated through the angle 180° (π/k, in general). Among these columns are those that are invariant under reflection through the center-plane of the column ($\mathbf{Z}_2(\kappa)$), those that are invariant under reflection through a plane containing the cylinder axis ($\mathbf{Z}_2(\tau)$) and those that are invariant under both reflections. See Figure 9.

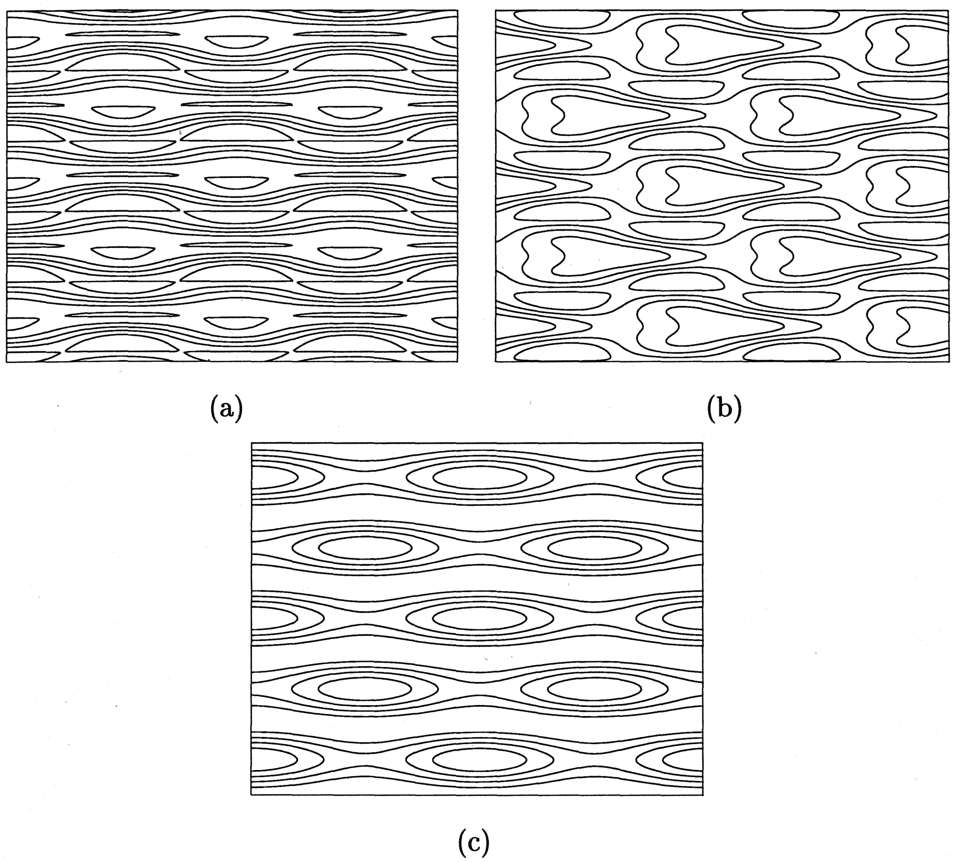


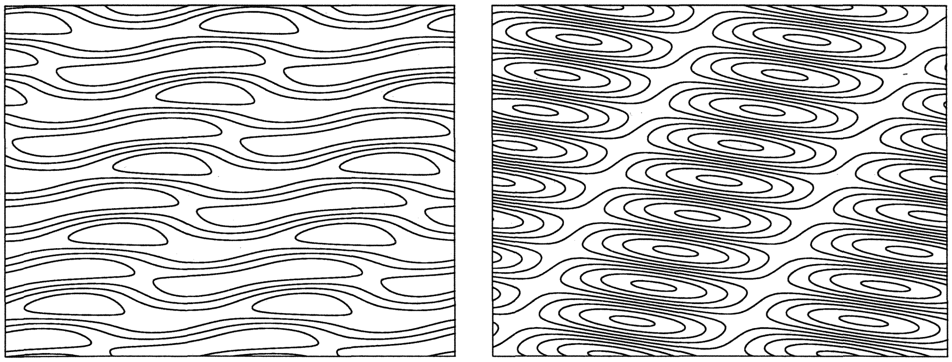
Figure 9: Corkscrew columns with $\pi/2$ rotation. (a) Left-right reflection; (b) Up-down reflection; (c) Left-right and up-down reflections.

There are two continuous families depending on ω with discrete corkscrew motions (those with $\mathbf{N}_\omega$ symmetry). See Figure 10.

3.3. Columns with Finite Symmetry — Seven Types

This types of column have neither a pure translation symmetry nor any symmetry that includes a translation symmetry. There are seven possible symmetry groups:

$$\mathbf{1} \quad <\kappa> \quad <\tilde{\kappa}> \quad <\tau\kappa> \quad <\tau> \quad <\tau,\kappa> \quad <\tau,\tilde{\kappa}> .$$



(a) (b)

Figure 10: Corkscrew columns with ω rotations where $0° \leq \omega \leq \frac{180°}{k}$. (a) No additional symmetry; (b) Up-down rotation.

An example of a column with no symmetry is given in Figure 11. Columns with just a single reflection or glide reflection are shown in Figure 12 while columns with exactly two reflection or glide reflection symmetries are shown in Figure 13.

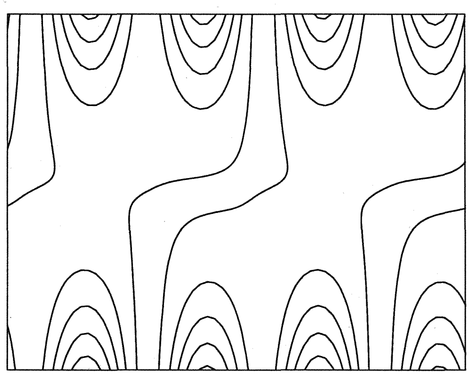


Figure 11: Column with no symmetries.

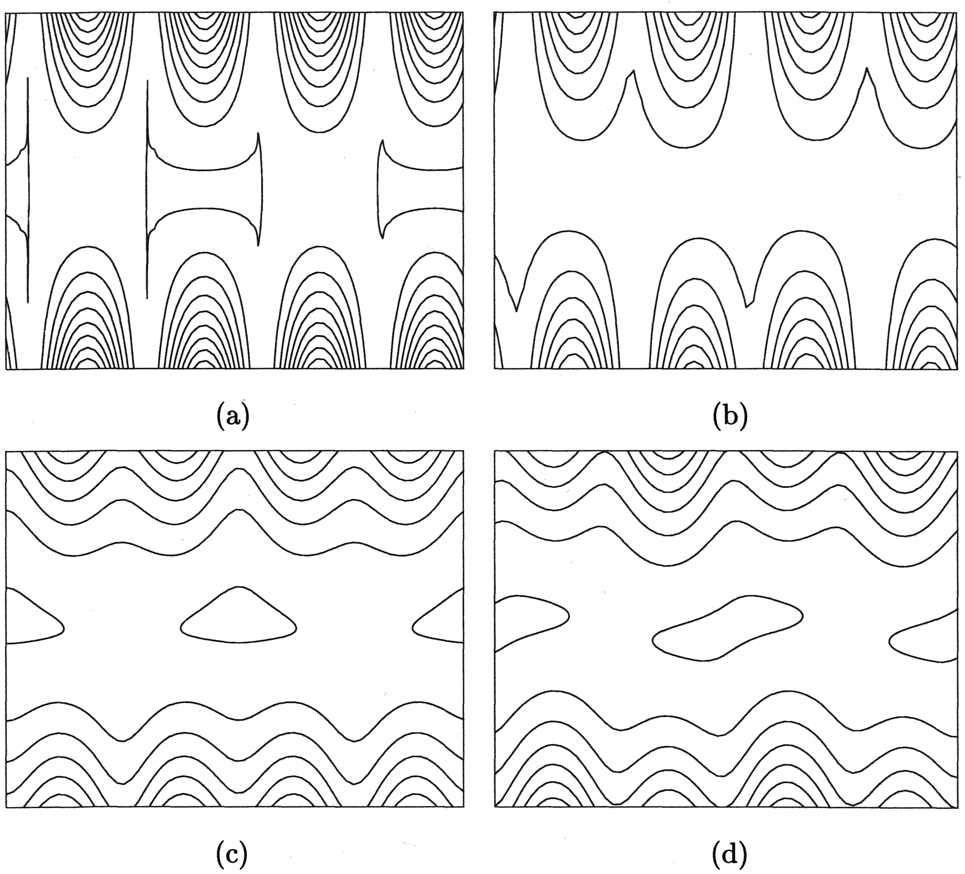
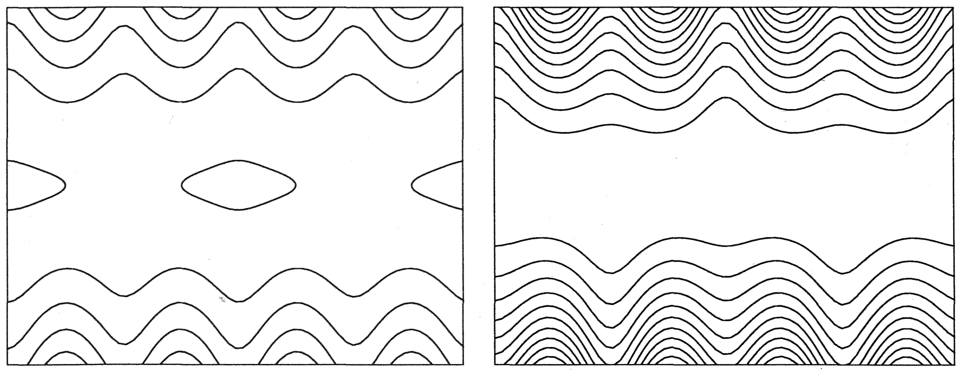


Figure 12: Columns with a single symmetry. (a) Up-down reflection; (b) Up-down glide; (c) Left-right symmetric; (d) Up-down rotation.



(a) (b)

Figure 13: Columns with two symmetries. Left-right reflection and: (a) Up-down reflection; (b) Up-down glide.

Acknowledgements: We would like to thank George Hersey for the suggestion that the study of columns by their symmetry might be interesting and worthwhile. Our discussion of symmetry and columns was a direct consequence of our participation in an interdisciplinary seminar on *The Biology of Beauty* as part of the symposium on *Order, Complexity and Beauty* organized by Tor Nørretranders at The Mindship in Copenhagen during the summer of 1996. MG is grateful for the opportunity to have participated in this seminar and for the support of interdisciplinary research that it provided.

We also thank Mike Field for pointing out the existence of the rod group classification. This research was supported in part by NSF Grant DMS-9704980.

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