Bridges
Mathematical Connections
in Art, Music, and Science

Pythagorean and Platonic Bridges between Geometry and Algebra

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Heuristic Introduction

Waldorf Schools (best known to U.S. math teachers through work of Hermann von Baravalle [1]) have continuing class-teachers who grow with their students from 1st to 8th grade, teaching almost everything (except foreign languages) in 1st and undergoing voluntary gradual diminishment from authority figures to friends by 8th grade (with ever more subjects left to specialty teachers). This is intended both to let ideas grow in natural progressions from year to year, with purposeful continuity, and to help ensure that behavioral problems (on part of both students and teachers) are dealt with and not merely passed on, thus helping to keep instruction lively (and life instructive).

But subject-specialized high school teachers (9-12) in such schools face the same difficulty that most teachers have at all levels in other schools: how to maintain freshness when teaching basically the same things (the same "material"), year in year out, for many years? One solution I found was to embrace the vagaries of scheduling and make it a challenge to teach things in different ways at different times of year, befitting the changing seasons. An autumn math class on the Pythagorean theorem might apply it to optimizing bird flight paths over warm land and cold water, while the same class in winter might start with crystal forms and then derive proofs of that theorem from tilings.

What follows are some things that have occurred to me as possible treatments of classical topics that would tie in either with hand-work lessons or the experiences of students who had been learning, say, about Celtic crosses in 9th grade art history, undergoing the usual year-long Euclidean geometry course in 10th, or being given a survey of astronomy in 11th. The last topic I have also developed further for master's-level courses in advanced Euclidean and non-Euclidean geometry, as part of secondary school teacher preparation at university.

Variations on the Pythagorean Theorem

Historical Background

The distance between any two points whose Cartesian coordinates are (x,y) and (s,t) is $\sqrt{(x-s)^2+(y-t)^2}$, which is familiar enough to teachers but apt to seem daunting to students on first encounter with its four-deep nested layers of algebraic operations to be managed in proper order.

It may be helpful first to go through the historical origin of Cartesian coordinates themselves, as a kind of recapitulation for individuals of what humanity has already experienced. The Egyptian word 𓐍𓂋𓏏, which may be transcribed *kh r t*, has survived into modern English in several forms: *chart*, *card*, *carton*, and *cartoon*, the latter meaning simply a big piece of card or carton. Michaelangelo prepared "cartoons" prior to executing the paintings on the ceiling of the Sistine chapel, i.e. he made hand-held sketches covered with square grids to permit their square-by-square enlargements to full size. This is exactly what the ancient Egyptians did, who originated the practice for transferring similar small sketches onto monumental walls, a square at a time, following in the process a set canon of beauty with 18 squares from ground to hairline during Old and Middle Kingdoms, 21 squares in New Kingdom Egypt and later adopted by Greeks [2]. With a smile, we can note the serendipity with which the 17th century French philosopher who independently reinstated this process for purposes of founding analytic geometry should have been named René Des*Cartes*.

Secondly, we can let one of the two points from which we are measuring distance be the origin (s,t) = (0,0) so that the distance to (x,y) becomes more simply $d = \sqrt{x^2+y^2}$, thereby eliminating one of the algebraic layers.

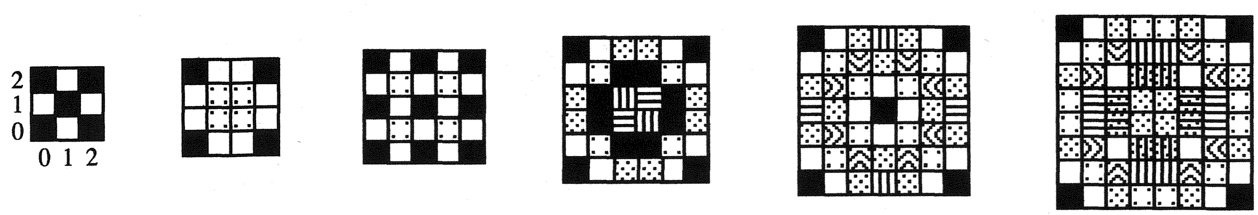
Thirdly we can eliminate the radical by squaring both sides, obtaining $d^2 = x^2 + y^2$ in which form the symbolism of the algebra has obliged by coming recognizably close to the usual language of the geometry, which traditionally refers to the squares on the two legs of a right triangle summing to the square on the hypotenuse.

A Modular Approach in 2 Dimensions

Modular or "clock" arithmetic is one of the numerical concepts taught in New Math, allowing us to express the fact that, say, 13, 14, and 15 o'clock military time are equivalent to 1, 2, and 3 o'clock civilian time because 13 = 12+1, 14 = 12+2, 15 = 12+3, and so on, hence $13 \equiv 1$, $14 \equiv 2$, $15 \equiv 3$, etc. when the modulus is suppressed by declaring $12 \equiv 0$; for by saying "$13 \equiv 1$ (mod 12)" we mean that we read a certain hand position as "1 o'clock" no matter how often the hour hand has been around that clock, i.e. $13 \equiv 12n + 1$ (mod 12) for any whole number n.

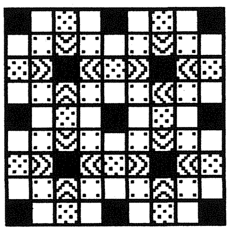
This modular approach, so familiar from *time*-keeping, can also be applied to *spatial* considerations in geometry lessons, with quite pleasing artistic results that can extend into craft projects such as quilt-making. Suppose we convene that the centers, rather than the corners, of the squares in a Cartesian grid are what is referred to by pairs of coordinates, so that (0,0) means the center of a central square and (x,y) means the center of a square that x units away from it horizontally and y units away vertically, restricting x and y to be integers. Then if we can agree on some color code for the integers from 0 to m -1 (mod m) we may color all squares of the grid according to the squares of their distances $d^2 = x^2 + y^2$ (mod m) and explore what sorts of patterns we find resulting. Let us take black for 0 and white for 1, for starters, and go on to convene that 2 be yellow, 3 be red, and 5 be blue, using primary colors for the first three prime numbers, then (honey-)orange for 6 = 2×3, green for 10 = 2×5, and (arbitrarily) violet for 7, with progressively dot-darkened versions of yellow for its higher powers 4 and 8, similarly darkened red for 9, brown for 11, and a cross for 12, which may be rendered for purposes of black and white illustration as follows: 0 ■, 1 □, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. That will allow us to survey the results for moduli $1 \le m \le 13$.

The resulting grid for mod m = 1 is all black (not shown), while that for m = 2 is a black and white checkerboard. Prime moduli m = 3, 5, and 7 yield central 2×2 squares of equal colors, while composite modulus m = 4 has colors for 0,1,2 equispaced but no 3 appearing because sums of squares can never yield a prime of form 4n-1 such as p = 3, 7 or 11, only primes of form 4n+1 such as 5 or 13 [3]. Further even moduli 6, 8, 10, and 12 (not shown) yield black central squares

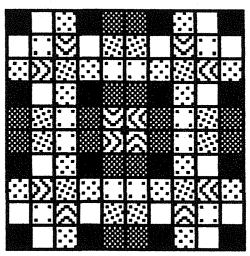


Mod 2 Mod 3 Mod 4 Mod 5 Mod 6 Mod 7

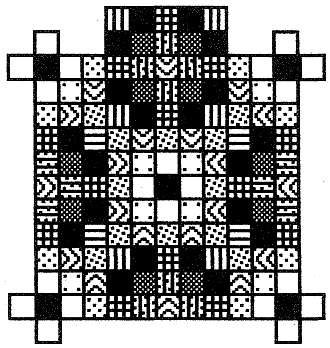
that behave like new origins, splitting their designs in halves; those which are multiples of 4 continue to omit primes of form 4n-1. Odd moduli m = 9 and 11 (not shown) continue to yield central 2×2 squares of equal colors. Modulus m = 10 also yields intriguing octagonal, quasi-circular, arrangements of black squares centered on midpoints of edges joining sucessive repeats of the actual origins, and similarly for mod m = 13 though its circles appear centered about common corners of 2×2 squares of equal colors. Besides drawing the missing cases for m = 11 and 12 not shown below, the interested reader may wish to try to formulate and prove algebraically (with



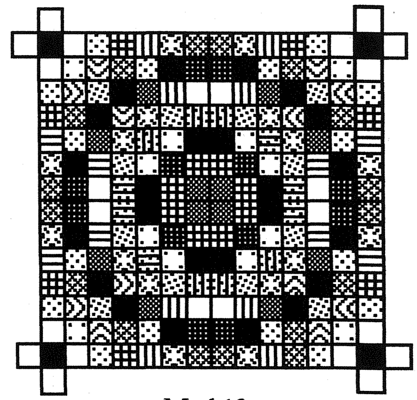
Mod 8



Mod 9



Mod 10

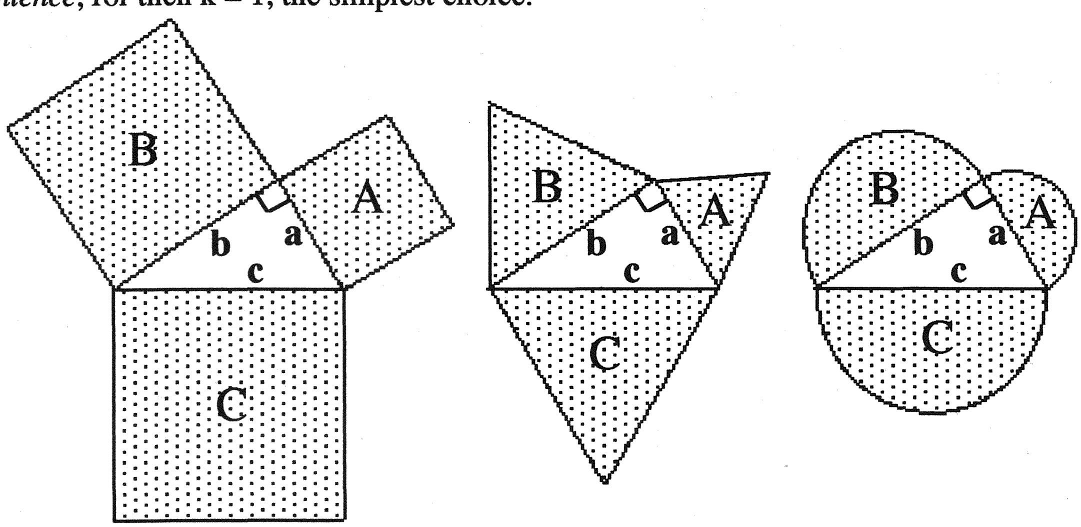


Mod 13

left brain) what the eye so easily observes geometrically (with right brain) as *theorems* about the circles in cases $m = 10$ and 13, for that word has the same Greek derivation as our word for *theater* where one goes to see a show, thereby establishing another Cartesian bridge between algebra and geometry (as well as between respective brain hemispheres). To do so, remember that circle centers at midpoints of grid squares for even moduli will have integral coordinates, while those centered at grid square corners for odd moduli must use half-integer coordinates.

Further Thoughts on Squares and Squaring

It is time to put the bridge we have begun to build between geometry and algebra to a first "stress test" and see how well it holds up. To what extent does the Pythagorean theorem actually have to do with *squares*, geometrically? Not at all! *Any* three similar figures could be used, such as equilateral triangles or semicircles (even Rorschach ink-blots, if properly sized), for in every case we have areas A + B = area C. If the similar figures are equilateral triangles with sides $s = a,b,c$, then their heights are $h = \frac{1}{2}s\sqrt{3}$ and areas $\frac{1}{2}sh$, so $A + B = C$ means $\frac{1}{2}a(\frac{1}{2}a\sqrt{3}) + \frac{1}{2}b(\frac{1}{2}b\sqrt{3}) = \frac{1}{2}c(\frac{1}{2}c\sqrt{3})$, or $\frac{1}{4}a^2\sqrt{3} + \frac{1}{4}b^2\sqrt{3} = \frac{1}{4}c^2\sqrt{3}$. If the figures are semicircles of diameters $d = a,b,c$, then their areas are $\frac{1}{2}\pi r^2 = \frac{1}{2}\pi(\frac{1}{2}d)^2 = \frac{1}{8}\pi d^2$, so $A + B = C$ means $\frac{1}{8}\pi a^2 + \frac{1}{8}\pi b^2 = \frac{1}{8}\pi c^2$. Either way, we obtain $k(a^2 + b^2 = c^2)$ for some scale factor $k = \frac{1}{4}\sqrt{3}$ or $\frac{1}{8}\pi$. The use of geometric squares to illustrate the theorem therefore turns out to be not a necessity but merely a matter of numerical *convenience*, for then $k = 1$, the simplest choice.



The matter becomes highly non-trivial when the question is posed the other way around: To what extent does the Pythagorean theorem actually have to do with *squaring*, algebraically? For this question, I am indebted to a former student at the High Mowing Waldorf School in

Wilton, N.H., Christopher Stoney, who asked me in the late 1960's whether it was merely convention or whether there were a deeper reason why we read 2nd and 3rd powers as "squared" and "cubed" and not, say, as "triangled" and "tetrahedroned." I was able to give a first approximation to an answer to this fairly quickly by analytically continuing the form of triangular and tetrahedral figurate numbers via the gamma function, and was able to show that in this sense any real power of any base could be defined in such a way, which made the answer seem to be again that the choice for squares and cubes was a matter of convenience.

The deeper answer did not come for another 14 years, at which time I was a graduate student at the University of Montana. There it occurred to me to ask whether there would exist coefficients in a Pascal-like array that would permit expansion of such generalized figurate powers according to an appropriately generalized binomial theorem. To my amazement, they do — for quite a wide variety of figurate forms — and the Pascal-like array in case of triangle- or simplex-type figures consists, appropriately simply, of all 1's.

For example, powers of 5 should be alike whether we partition it as 1+4 or 2+3. So if the numbers $n = 1$ to 5 have squares $n^2 = 1,4,9,16,25$, cubes $n^3 = 1,8,27,64,125$, triangles $\frac{1}{2}n(n+1) = 1,3,6,10,15$, and tetrahedra $\frac{1}{6}n(n+1)(n+2) = 1,4,10,20,35$, then just as we are able to use the familiar 2nd and 3rd rows 1 2 1 and 1 3 3 1 of the Pascal triangle to expand $(1+4)_{squ}$ or $(1+4)^2 = \mathbf{1}\cdot1\cdot1 + \mathbf{2}\cdot1\cdot4 + \mathbf{1}\cdot1\cdot16 = 25$ agreeing with $(2+3)_{squ}$ or $(2+3)^2 = \mathbf{1}\cdot4\cdot1 + \mathbf{2}\cdot2\cdot3 + \mathbf{1}\cdot1\cdot9 = 25$ and $(1+4)_{cub}$ or $(1+4)^3 = \mathbf{1}\cdot1\cdot1 + \mathbf{3}\cdot1\cdot4 + \mathbf{3}\cdot1\cdot16 + \mathbf{1}\cdot1\cdot64 = 125$ agreeing with $(2+3)_{cub}$ or $(2+3)^3 = \mathbf{1}\cdot8\cdot1 + \mathbf{3}\cdot4\cdot3 + \mathbf{3}\cdot2\cdot9 + \mathbf{1}\cdot1\cdot27 = 125$, so we may also use the all-unit rows 1 1 1 and 1 1 1 1 of the alternative simplex-type array to expand $(1+4)_{tri} = \mathbf{1}\cdot1\cdot1 + \mathbf{1}\cdot1\cdot4 + \mathbf{1}\cdot1\cdot10 = 15$ agreeing with $(2+3)_{tri} = \mathbf{1}\cdot3\cdot1 + \mathbf{1}\cdot2\cdot3 + \mathbf{1}\cdot1\cdot6 = 15$ and $(1+4)_{tet} = \mathbf{1}\cdot1\cdot1 + \mathbf{1}\cdot1\cdot4 + \mathbf{1}\cdot1\cdot10 + \mathbf{1}\cdot1\cdot20 = 35$ agreeing with $(2+3)_{tet} = \mathbf{1}\cdot4\cdot1 + \mathbf{1}\cdot3\cdot3 + \mathbf{1}\cdot2\cdot6 + \mathbf{1}\cdot1\cdot10 = 35$.

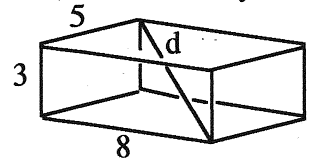
The hard work then lay in passage to the DeMoivre limit and deriving the analogue of Gaussian normal or bell-curve distribution. This yielded the *arcsine* distribution otherwise known as "gambler's ruin" [4] which is just what one should have expected, for the historical origin of the usual array (as rediscovered by Pascal — the Chinese had had it much earlier) was a request on the part of members of the French aristocracy to work out fair betting odds for them, since they had been losing much money by betting e.g. that 0, 1, or 2 heads were equally likely (probability $\frac{1}{3}$) when tossing 2 coins, which is precisely the all-1's distribution. He showed that the true odds were 1:2:1 (probabilities $\frac{1}{4}:\frac{1}{2}:\frac{1}{4}$), not 1:1:1 ($\frac{1}{3}:\frac{1}{3}:\frac{1}{3}$), and then sadly fell off his horse in a riding accident, assumed it was divine punishment for having aided sinners, and foreswore further work in mathematics.

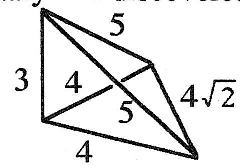
I was then able to return to Christopher's original question and give it a deeper answer: The choice of square, cube, and general measure polytope family of figures for expressing powers is, after all, not a matter of convenience but *necessity*, numerically, for then and only then is a power of a product equal to a product of powers, giving power distribution $(ab)^n = a^n b^n$, without which *prime power decomposition* would not be possible [5].

The Impossible and Possible Extensions to 3 or More Dimensions

Fermat's last theorem (i.e. his last conjecture to be settled posthumously), which states that $x^n + y^n = z^n$ has solutions in integers x,y,z if and only if n is a positive integer less than 3, has recently been verified by Andrew Wiles [6]. Why did this fact take so long and require such recondite methods to prove? Because, I suggest, it was in a sense the "wrong way 'round" to seek extensions of the Pythagorean theorem in terms of higher powers of lengths, whereas easy extensions are available in terms either of squares of more than 2 lengths or squares of 2- or more-dimensional contents such as areas or volumes. The former is more familiar: Suppose a rectangular solid has side lengths 3, 5, and 8; then the length of its space diagonal is found via d =

$\sqrt{3^2+5^2+8^2} = \sqrt{98}$ or $7\sqrt{2}$, generalizing the former $d = \sqrt{x^2+y^2}$ to $\sqrt{x^2+y^2+z^2}$ in terms of lengths x,y,z. Less well-known (in fact virtually unknown, though quite elementary — I discovered it unaided in





10th grade) is the latter, of which an example might be a tetrahedron with three edges of lengths 3,4,4 meeting at mutual right angles, whose three leg faces have squared areas summing to the square of the area of the hypotenuse face with edges $5,5,4\sqrt{2}$, as easily verified for this particular case to yield $6^2 + 6^2 + 8^2 = 136 = (2\sqrt{34})^2$. A general proof (tedious but straight-forward, accessible to an average high school student) may be had via Heron's formula $A^2 = s(s-a)(s-b)(s-c)$ for the square of the area of a triangle with side lengths a,b,c, where $s = \frac{1}{2}(a+b+c)$.

The significance of the latter version is that it can provide a first intuitive approach to the otherwise very notation-obscured topic of *tensors*, extending the notion of vectors as directed lengths to directed areas and other higher-dimensional contents. The notion of hypotenuse and leg areas with given orientations can be applied, for example, to the action of wind in a sail, analyzed (for tacking purposes) as composed of actions in cardinal directions [7]. It should then come as no surprise to learn that sailing was the sport of choice of Albert Einstein, who decided to leave Germany when the Berlin authorities confiscated his boat.

Variations on Platonic Solids

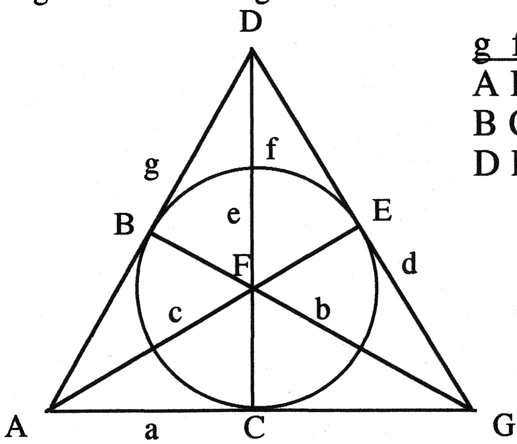
Finite Projective Planes over Primes

If the usual pair (x,y) of coordinates of a point in the Cartesian plane is embedded into the triple (x,y,1) with unit in third position and then permitted to be resized by any non-0 scale factor k as (kx,ky,k) = (x',y',z'), the resulting three numbers are called *homogeneous coordinates* of a point in the projective plane. Moreover, Cartesian equations of lines such as $ax+by+c = 0$ can be abbreviated as [a,b,c] to yield dual homogeneous coordinates of a line in the projective plane, including [0,0,1] as line at infinity, dual to (0,0,1) as origin. A line [a,b,c] is then incident with a point (x,y,z) if and only if their dot product $[a,b,c]\cdot(x,y,z) = ax+by+cz = 0$, while the cross product of any two points yields their joining line and, dually, the cross product of any two lines yields their point of intersection [8].

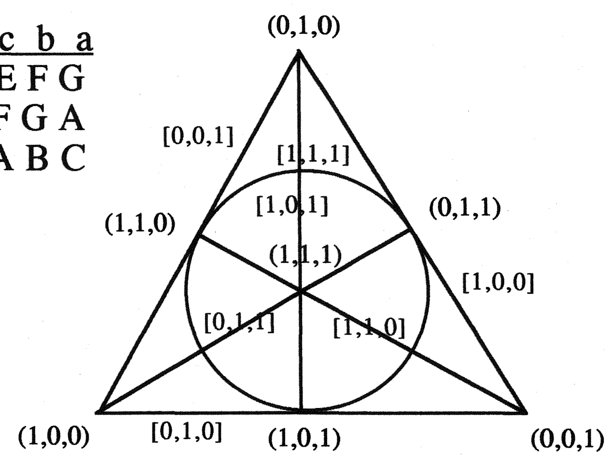
Since the all-zero triples (0,0,0) and [0,0,0] must be excluded (lest they be incident with everything, anomalously), it follows that for any prime p there will be p^3-1 possible triples of coordinates over a finite field of integers mod p. But since such triples are only determined up to proportionality and there are $p-1$ possible non-zero scalars k to multiply them by, there can only be $(p^3-1)/(p-1) = p^2+p+1$ distinct points and lines in any such finite projective plane.

The first three primes p = 2, 3, and 5 therefore yield finite projective planes with 7, 13, and 31 points and lines, respectively. But these are just the numbers of symmetry axes of the five regular solids, as described in Plato's *Timæus*: The tetrahedron has 4 pairs of face planes and corner points + 3 pairs of opposite edges, totalling 7 axes; the cube has 3 pairs of faces + 6 pairs of edges + 4 pairs of corners, totalling 13 axes (the octahedron simply interchanges the roles of faces and corners); and the pentagon dodecahedron has 6 pairs of faces + 15 pairs of edges + 10 pairs of corners, totalling 31 axes (the icosahedron again interchanging roles of faces and corners). This is such a suggestive result, one would expect to find it dealt with in most texts on related subjects; instead, while "well known to those who well know such things" (as Richard Guy likes to quip), it is scarcely to be found in the formal literature [9]. The reason for the common numbers, it turns out, is that the groups of symmetry motions of the regular solids are subgroups of the groups of collineations of the respective finite planes, a face axis being different from an edge axis of a regular solid but all points of a projective plane being alike, so the latter has more symmetries than the former.

The simplest case is the 7-point plane, also known as Fano plane after its discoverer in 1892 (Coxeter [8], p. 91), coordinatized using only 0's and 1's mod p = 2. As schematically illustrated at right below in terms of these coordinates, it would seem that points (1,0,0), (0,1,0), and (0,0,1) played a distinguished role as "corners" of a "triangle" with (1,1,1) as "center," but these words involve metric notions that are inappropriate in the finite context. The p+1 = 3 points incident with any given line are the *only* points of or on that line — the graphic device of connecting them by continuous strokes is just that: a graphic device; the apparent intermediate points along those connecting strokes are not actually part of the configuration which has but 7 points and



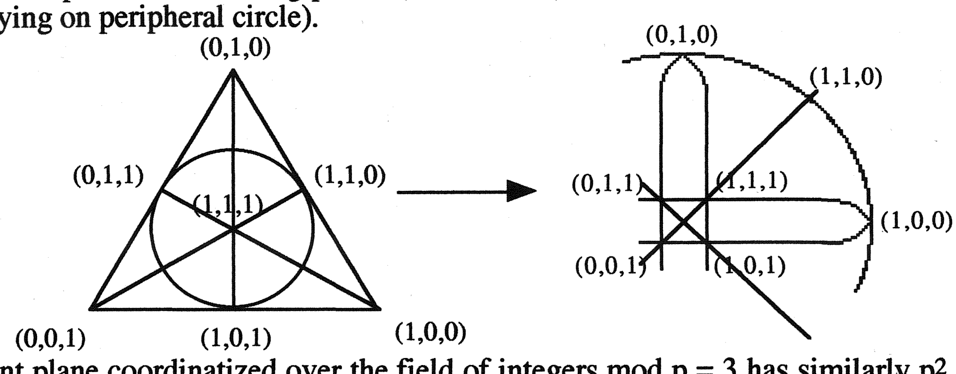
g	f	e	d	c	b	a
A	B	C	D	E	F	G
B	C	D	E	F	G	A
D	E	F	G	A	B	C



7 lines. Calling them A to G and a to g, respectively, as at left above helps avoid the temptation to suppose there to be more elements than are actually present. Moreover, the incidence table of those 7 pairs of elements (the columns headed with line names, listing the 3 points incident with each, whose dual would have lower and upper case letters reversed) shows clearly that all 7 elements are completely alike, cyclic as shown, with no distinguished first or last element.

The apparently special appearance of the circular-seeming line [1,1,1] disappears when called f and placed in smoothly cyclic context of the table. The figure may be renamed in many ways with any one of the 7 lines playing that apparently circular role (the set of all ways to do so being precisely the set of collineations of the plane).

On the other hand, if the apparently special role of that circular-appearing line is emphasized by reproportioning the figure so that seeming circle grows in size (to our eyes) and tends toward the infinitely-distant ideal line of the plane, then the connection of the projective with the Euclidean plane becomes clear: The latter first distinguishes and then deletes that ideal line, declaring any remaining lines tending to meet in a point of it to be "parallel." As way-station in that process of passage from projective to Euclidean, we may recognize Celtic circle-crosses. The $p^2 = 4$ points in center of such a cross bear recognizable Cartesian coordinates (when final 1 is deleted), while the p+1 = 3 remaining points (with final 0) are "ideal," unreachable, at infinity (symbolized by lying on peripheral circle).



(The 13-point plane coordinatized over the field of integers mod p = 3 has similarly $p^2 = 9$ "finite" points in a square and p+1 = 4 remaining points on the circular-appearing "line at infinity," as schematized on cover of Dorwart [8]).

Error Detecting and Correcting Codes

If we now use 0's and 1's a different way as part of a 7×7 incidence matrix, writing 1 in the ij^{th} place if point i is incident with line j in the Fano plane and 0 otherwise, we obtain seven 7-digit strings consisting of four 0's and three 1's in regularly shifted locations. If any of these entries should happen to be switched (exchanging a 0 for a 1 or vice versa) the error would be immediately detected (by change in parity of sum from odd 3 to even 2 or 4); moreover, there would be a unique choice of which single entry to correct (thanks to regularity of locations) in order to obtain an allowed string. Such schemes, based on finite projective planes and other highly symmetric patterns, became applied to detecting and correcting errors in noisy electronic signal channels in the 1950's and 60's. When the Voyager probes were sent to Jupiter and Saturn in the 1970's, highly redundant codes were used for safety, based on larger regular solids; when they went on to Uranus and Nepture in the 1980's, to save energy, they were reprogrammed to use smaller ones, with excellent results. What appear to us as "color photographs" taken by Voyager are in fact highly computer-processed combinations of three different black-and-white television images (one each, through three color filters). Each television image consists of lines scanned from top to bottom of screen, and each line consists of individual pixels of varying brightness or dimness. The Platonically-based codes turned these shades of gray, in effect, into well-distinguishable (since regularly placed) "points" of a highly symmetrical geometric scheme for purposes of safe transmission of extremely feeble voltages across extremely wide distances; the computers at Jet Propulsion Lab in Pasadena turned them back into shades of gray again (not only detecting and correcting random transmission errors but also correcting for systematic distortions such as changes in perspective in the course of a rapid fly-by, the top and bottom portions of any apparent single "picture" being in fact taken from appreciably different view-points in space and time).

Returning to the historical background of such modern mathematics in the classics, it is touching to read Plato's account of the death of Socrates who, having taken the hemlock, describes feeling himself grow distant from the Earth, seeing it as looking like "one of those balls of multicolored leather" that the boys liked to kick around on their *gymnasium* fields — the ancestor of our modern soccer ball, with just the symmetries that would one day be used to grow distant from the Earth by proxy, via Voyager, and see the rest of our solar system by electronic extension of human sight.

Notes and Bibliography

[1] Hermann von Baravalle. Numerous articles by this master teacher may be found in 1940's and 50's issues of *The Arithmetic Teacher*, *The Mathematics Teacher*, and *Bulletin of the Yeshiva Institute*. His geometrical work, as taught at Adelphi, was collected by Col. Beard and has been re-issued by Creative Publications.

[2] See Erik Iversen in collaboration with Yoshiaki Shibata, *Canon and Proportions in Egyptian Art*, Aris and Phillips Ltd., Warminster, 1975, or Gay Robins, *Proportion and Style in Ancient Egyptian Art*, Univ. of Texas Press, Austin, 1994.

[3] See G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, pp. 299-300, 4th edition, Oxford, 1968, or B.M. Stewart, *Theory of Numbers*, pp. 192-194, 2nd edition, New York, 1971.

[4] Rarely discussed even in advanced texts, one treatment may be found in William Feller, *An Introduction to Probability Theory and its Applications*, Vol. I, Fig. III-3 of 3rd edition (only!), John Wiley & Sons, New York, 1968.

[5] Stephen Eberhart, *Figurate Powers*, Research Note 22, Dept. of Math., Univ. of Montana, Dec. 1983.

[6] Simon Singh, *Fermat's Enigma*, Walker and Co., New York, 1997.

[7] Lillian R. and Hugh Gray Lieber, *The Einstein Theory of Relativity*, pp. 131-139, Holt, Rinehart & Winston, 1936 & 1945. The illustration on p. 136 shows implicit equivalence of these two theorems: If tetrahedron is OABC, with OG the normal from O to ABC and OK the projection of OG onto OA, then length ratio OK/OG = area ratio OBC/ABC, since both = $\cos\angle GOK$ = cosine of dihedral $\angle$ between OBC & ABC.

[8] See Harold Dorwart, The Geometry of Incidence, Chapter II, Autotelic Instructional Materials, New Haven, 1966, or H.S.M. Coxeter, *Projective Geometry*, pp. 112-115, Univ. of Toronto Press, 1974.

[9] I am aware only of a series of in-house publications by Fernand Lemay of the Laboratoire de Didactique, Faculté des Sciences de l'Éducation, Univ. Laval, Québec, in particular those collectively titled *Genèse de la géométrie* I - X.