BRIDGES
Mathematical Connections
in Art, Music, and Science

Architecture and Mathematics: Art, Music and Science

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Introduction

It is a great pleasure to write a paper about architecture and mathematics on the occasion of the conference, *Bridges: Mathematical Connections in Art, Music and Science*. It is architecture's intimate relationship to mathematics that underscores its ties to art, music and science. The subject is too vast to lie within the range of a single discussion; this paper will look at some facets of these various relationships with the aim of introducing the reader to ideas meriting further study.

We are surrounded by architecture. It determines the myriad spaces within which our lives unfold: where we live, work, worship; where we are taught, where we are healed, and, eventually, where we are buried. On one level, the relationship between mathematics and architecture seems a practical one: architecture has dimension and can be measured, therefore, it relates to numbers. It has shape and volume, as do plane figures and solids, therefore it relates to geometry. It involves composition and relationships, and therefore has to do with ratio, proportion and symmetry.

One essential quality of architecture is that it must be constructed. This means that the architect must be capable of describing the building and explaining how it is to be built. In ancient times this was no mean feat: many people believe that ancient architects were often members of secret societies and were forbidden under threat of death to divulge secrets of the trade to outsiders, so workers were told only the most superficial details of what they were constructing. Additionally, workers were more often than not illiterate. It is known that in times as recent as 500 years ago architects merely explained their designs to workers, never producing the exhaustive set of construction documents that today's architects labor over. How were the designs explained?

Very often they were explained through geometry. One key dimension was laid out on the site, and other dimensions determined by means of geometric constructions, often with strings, ropes or wires, but sometimes with actual giant compasses.

It has been argued, however, that measuring length and width has very little to do with pure mathematics, that the act of counting must not be confused with the science of number theory [1]. Pure mathematics has to do with ideas, not with measurable quantities. So how is one to describe the relationship between architecture and mathematics?

Architecture and Mathematics

Much light may be shed on the subject by considering the Roman architect Vitruvius' answer to the question, What makes architecture? Vitruvius wrote in the *Ten Books on Architecture* that good architecture must have three qualities: firmness, commodity, and delight [2]. In "firmness", we may understand that he intends structural integrity: the building must stand up. It is in structures that a relationship of architecture to science is found. It is in Vitruvius' quality of "delight" that we find the relationship of architecture to art and to music (not in the sense of music that we hear, but in another sense, as we shall see). But the fundamental relationship between architecture and mathematics is found in the meaning of "commodity." Commodity may be understood in terms of utility, but also in terms of "fitness," that is, the building must fit the function. One important function of architecture is to impart meaning to our lives, a function it accomplishes through symbolism. Here is where the relationship of architecture and mathematics is understood in its cultural context.

Philosophically, mathematics is considered as "pure" thought, the closest a human being can come to the divine. This is why architects very often employ mathematical ideas in their designs, in order to give expression to cultural beliefs about the cosmos and the Creator, about society and nation, about humanity and the natural world. An example of the use of mathematics to form express cultural beliefs in architecture is found in the Pantheon of Rome, built early in the second century A.D. (ca.117-128). The Pantheon was a temple dedicated to all gods, and was intended to be a model of the cosmos. It was also intended to express the greatness of the Roman empire. The interior is composed of a semi-circular dome resting on a cylinder of equal diameter (figure 1). Because the rotunda is as high as it is broad, a perfect sphere could be inscribed within its space. The sphere may, in its turn, be inscribed in a cube (figure 2)[3]. The pure geometric forms express a rich symbolism: the perfect sphere of the volume represents.

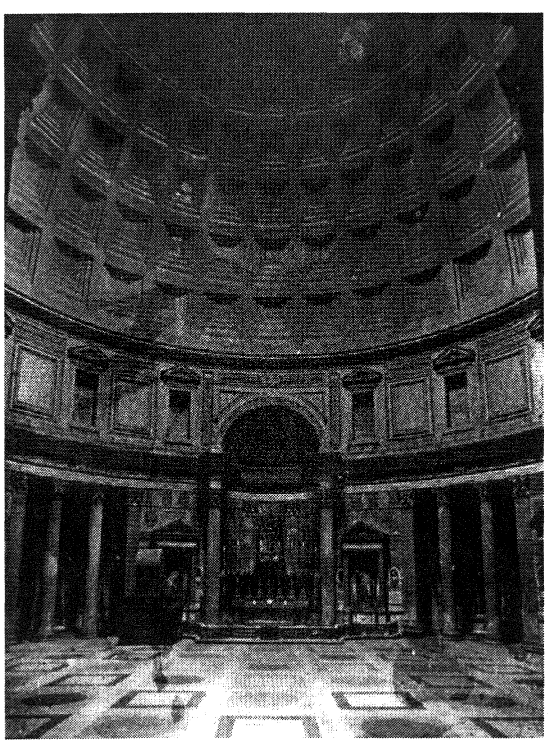


Figure 1

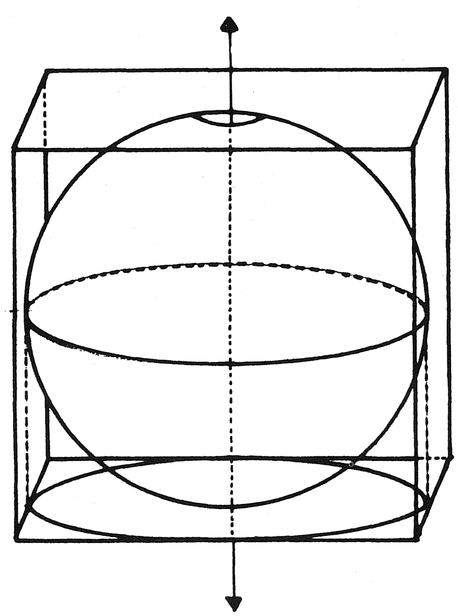


Figure 2

the cosmos; the perfect circle of the plan represents the terrestrial domain of the Roman empire. There is a reference made to the problem of squaring the circle, which represented the transformation of the earthly into the divine. One particularly evocative theory even grants a symbolic role to the convex surface of the pavement, claiming that the diameter of the rotunda was a scale representation of the extent of the Roman empire, and that the convex surface represents the curvature of the earth's surface over that area [4]. Thus, a mathematical vocabulary performs two functions in the Pantheon: it provides a basis for the architectural forms, and it imparts symbolic significance to the building.

Art and Architecture

In the Renaissance, many of the greatest artists were also architects, such as Brunelleschi and Michelangelo; not a few were also mathematicians, such as Alberti and Leonardo. It was natural in this period that art, architecture and mathematics should overlap. One of the greatest Renaissance discoveries in painting was that of perspective. Perspective allowed the artist to control the picture space, that is to determine the location of the viewer relative to the object viewed. Artist and mathematician Piero della Francesca taught the technique of perspective not as a visual exercise, but rather as a geometrical exercise [5]. He asserted the correctness of his method for constructing a square in perspective by providing proofs for theorems positing the equality of certain lengths (Figure 3).

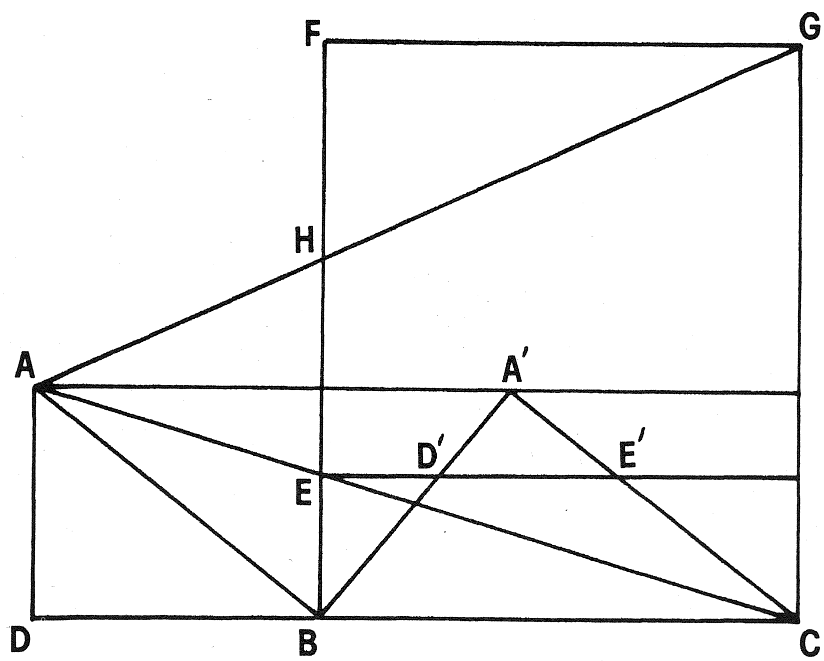


Figure 3

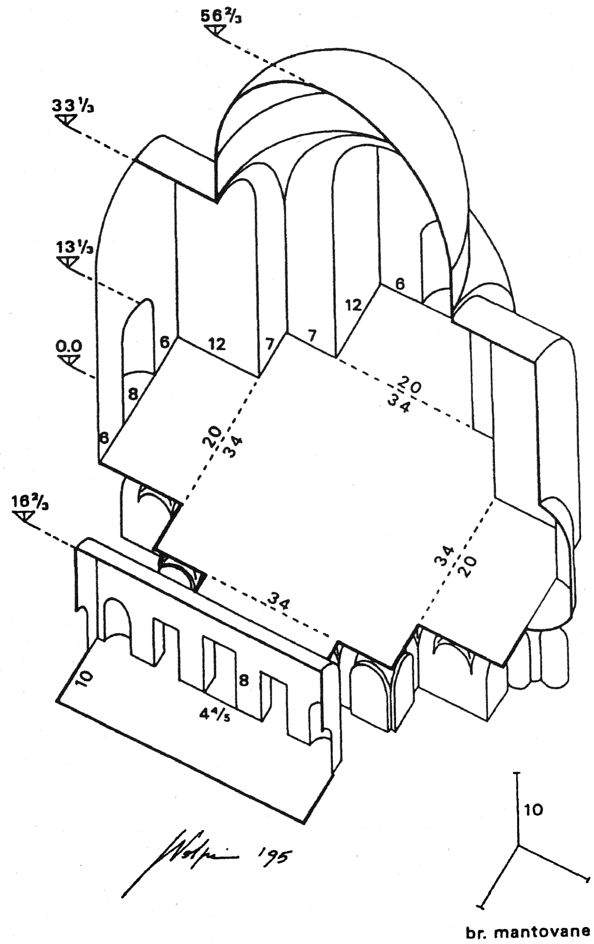
In his treatise on painting, *De Prospectiva Pingendi*, Piero led his readers from the construction in perspective of a square lying flat on the ground to tilings within the original square, as for example in a pavement pattern, to construction of prisms over the square to represent structures. Piero practiced what he preached, carefully depicted architectural scenes in his paintings. The depiction of architecture was not merely a matter of taste, however: architecture provided the context within which Piero could introduce explicit mathematical references [6]. (Hidden references to mathematics were contained in the geometric constructions which underlay the composition of the paintings [7].) One such reference is found in the pavement pattern depicted in Piero's painting *The Flagellation of Christ*. Extremely fore-shortened eight-pointed star octagons, symbolizing immortality and resurrection, appear in the bays directly in front of and behind Christ, while He stands in the middle of a circle, symbol of divinity. This is another example of a cultural use of mathematical forms. The use of architecture in painting also allows the artist to signal the location of the vanishing point. Thus, in Leonardo's *Last Supper* it is possible, by following the converging lines of the coffers in the ceiling, to ascertain the location of the vanishing point over the head of Christ. The vanishing point represented God, thus its location in the painting is of utmost importance.

Most importantly, the representation of architectural details permits the artist to represent the picture space mathematically. Art historian Erwin Panofsky noted that through the use of a floor pattern, even such a simple one as the checkerboard, the artist creates a coordinate system by which the location of objects and the distance between them may be quantified and compared by merely counting the tiles [8]. Thus, an architectural element permits us to know the precise position of Christ within the picture space. This is of course one of the roles a pavement pattern plays within an actual architectural space, as well as in the depiction of a space. Finally, the use of architecture in a painting allowed the artist to introduce an explicit system of significant proportions. In the *Flagellation of Christ*, for example, we can see that the architectural bays are perfect squares. Such proportions were references to the humanistic, neo-Platonic philosophy current in the Renaissance.

Music and Architecture

Because the art of architectural design has much in common with the composition of music, architecture has been described as "frozen music." One source for this metaphor is the story of the Greek god Amphion, son of Zeus, who built the walls of Thebes by playing his lyre, casting a spell over the beams and stones so that they arranged themselves according to harmonious proportions[9]. Ever since Pythagoras devised his musical scale as a means of establishing consonance and dissonance, similar means for establishing order in architecture have been sought. Architects have sometimes used the musical proportions, based on the ratios of integers 1, 2, 3 and 4 (and in some ages, 5) to introduce into their designs the kind of harmonic relationships found in music. Some critics of this practice have objected to the appearance of dissonant ratios, such as 4:5, but the use of such ratios seems to prove that the intent was not to create literally harmonic architecture, but rather to introduce order and coherence into the design [10].

An example of this is found in Leon Battista Alberti's design for the church of S. Sebastiano in Mantua. Livio Volpi Ghirardini's very detailed analysis of the dimensions in this building has revealed that they may be arranged in different numerical sequences based on the ratio of 3:5 [11]. The use of a single ratio to generate individual groups of relationships insures "harmony" among parts and a high degree of internal order (figure 4).



Central square	34, (34), 56 2/3
Arm	12, 20, 33 1/3
Portico	..., 10, 16 2/3
Entrance and apse	4 4/5, 8, 13 1/3

Figure 4

Another example of the use of musical ratios appears in Michelangelo's New Sacristy in the basilica of San Lorenzo in Florence [12]. This case is particularly interesting because the rational musical proportions are generated from a geometric construction based on the irrational square root of 2. Michelangelo's system begins with a cube describing the volume of the lower part of the chapel. From the cube is derived the root-2 rectangle which describes the Sacristy's groundplan (figure 5); other dimensions used in the architecture are determined through successive subdivisions of the root-2 rectangle (figure 6). Again, the use of such a system introduces a high degree of order into the composition.

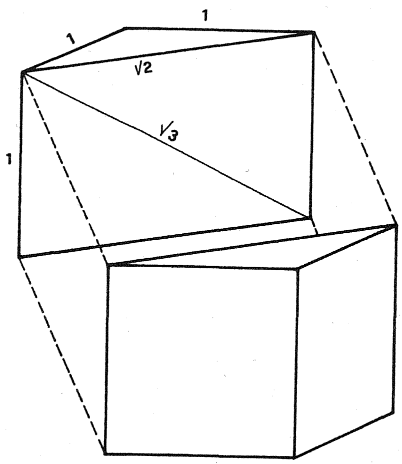


Figure 5

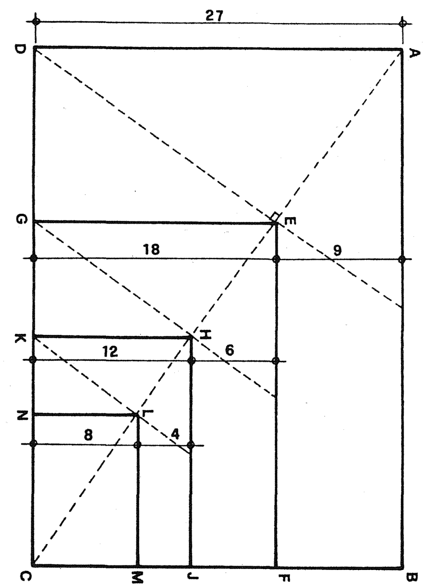


Figure 6

Michelangelo's use of the musical ratios and a geometric construction helps place him firmly in the context of his times. Several architectural treatises were written in the fifteenth and sixteenth centuries which described music and geometry in great detail. Perhaps the most important of these was Alberti's *Ten Books of Architecture*, written in about 1452 [13]. Architects Sebastiano Serlio and Andrea Palladio also wrote treatises on architecture, in which they recommended various systems of musical proportions for use in architecture [14].

Science and Architecture

In addition to their symbolic function, geometrical forms have played an important part in architecture in terms of the science of structure. Ancient architects learned that certain geometrical forms are endowed with structural integrity, that is, they are rigid. The example of this *par excellence* is the triangle. Because the triangle is a shape that retains its form even under applied pressure, architects from the age of ancient Egypt onwards have used triangle forms in critical locations. One might think of the pyramids, prisms formed of four triangles arranged about a square base, canting in so that their apexes meet at one point. Another use of the triangle is in the pediment of a Greek temple, an application where the rigidity inherent in the triangle aids in supporting the roof structure. It is said that when thirteenth century architects were having structural problems with the Gothic cathedral of Milan, consultants called in from Northern Europe recommended that the structure be made to conform to either the outline of an equilateral triangle or to a square, as these were believed at the time to be the two most structurally secure shapes. Modern structural engineers have developed many different forms of trusses, all of which are based on the same principle of the rigidity of triangles [15] (figure 7). Twentieth century architect Buckminster Fuller developed his geodesic dome based on the rigidity of triangles.

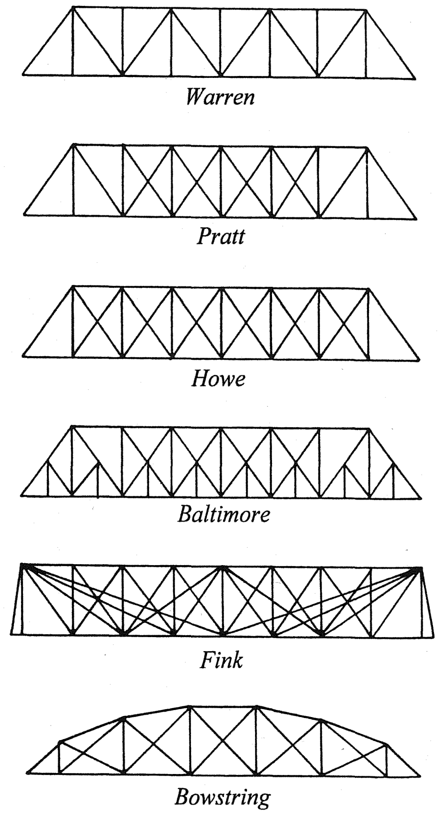


Figure 7

There is another intriguing sense in which architecture and science are related, however, that goes beyond the application of mathematics to solve structural problems. Mathematical ideas developed in the service of science have very often influenced architectural form on an aesthetic level. John Clagett has convincingly traced the relationship between architecture and science in the seventeenth and eighteenth centuries, arguing that the phenomenon of dynamics described by Newton and the formulation of analytic geometry by Descartes paved the way for the development of the architectural forms found in the central European Baroque churches [16]. Alberto Pérez-Gómez has described how seventeenth century French architects Claude and Charles Perrault, taking their inspiration from the development of positivistic science, set about trying to transform architectural theory so that it was ever more similar to mathematical reason [17]. Their aim was to transform architecture from a subjective art into an objective one, one that conveyed objective, mathematical truth rather than mystical ideas about the cosmos.

In conclusion

I hope that these considerations have cast some light on the way in which architecture, through its relationship to mathematics, is related to art, music and science. Engineer Mario Salvadori summed it up best when he said, "the relationships between mathematics and architecture are so many and so important that, if mathematics had not been invented, architects would have had to invent it themselves" [18].

Notes

[1] Mario Salvadori, "Can there be any relationships between Mathematics and Architecture," Nexus: Architecture and Mathematics, Kim Williams, ed. (Fucecchio, Florence: Edizioni dell'Erba, 1996).

[2] Cf. Vitruvius, The Ten Books on Architecture, Morris Hicky Morgan, trans. (New York: Dover Publications, Inc., 1960) book I, chap. III, sect. 2, 17. In this translation, "firmness, commodity and delight" are rendered "durability, convenience and beauty."

[3] Cf. William MacDonald, The Pantheon: Design, Meaning and Prodigy (Cambridge: Harvard University Press, 1976), figure 33.

[4] Cf. Gert Sperling, Das Pantheon in Rom: Abbild und Mass des Kosmos (Munich: Ars Una Verlag, 1998). In print.

[5] For a discussion of Piero's perspective technique as geometric construction, cf. Mark A. Peterson, "The Geometry of Piero della Francesca," The Mathematical Intelligencer 19, no. 3 (spring 1997), 33-40.

[6] Cf. Kim Williams, "What Piero Painted" (Letter to the Editor), The Mathematical Intelligencer, 20, 2 (1998).

[7] For a discussion of the geometry underlying the painting, cf. Martin Kemp, The Science of Art (New Haven: Yale University Press, 1998), 30-32.

[8] Cf. Erwin Panofsky, La Prospettiva come `Forma Simbolica' ed Altri Scritti, Guido Di Neri, editor (Milan: Feltrinelli Editore, 1960) 62.

[9] This myth is illustrated in a marble relief in the large hall of the Royal Palace at Amsterdam. I am grateful to Dirk Jacob Jansen of the Letterenbibliotheek in Utrecht for the reference. Later references to architecture as frozen music are found in literature. In *Corinne* (bk. IV, chap. 3) written in 1807, Madame de Stael writes, "The sight of such a monument is like a continuous and stationary music." Goethe wrote in a letter of 1829, "I call architecture frozen music." Schilling wrote in *Philosophy of Art*, "architecture in general is frozen music. My thanks to Ralph Lieberman and Peggy Davis for these references.

[10] For good descriptions of musical theory in architecture, cf. Rudolf Wittkower, Architectural Principles in the Age of Humanism (New York: W.W. Norton and Co., 1971), 101ff.

[11] Livio Volpi Ghirardini, "The Numberable Architecture of Leon Battista Alberti as a Universal Sign of Order and Harmony," Nexus: Architecture and Mathematics, Kim Williams, ed. (Fucecchio, Florence: Edizioni dell'Erba, 1996), 147-166.

[12] Cf. Kim Williams, "Michelangelo's Medici Chapel: The Cube, the Square and the Root-2 Rectangle," Leonardo 30, no. 2 (1997), 105-112.

[13] For Alberti's discussions of musical proportions and geometry, cf. Leon Battista Alberti, The Ten Books of Architecture, 1755, reprint, (New York: Dover Publications Inc., 1986), book IX, chaps. V and VI, 194-201.

[14] Sebastiano Serlio, The Five Books of Architecture, 1611, reprint, (New York: Dover Publications Inc., 1982); Andrea Palladio, I quattro libri dell'architettura (The Four Books of Architecture), trans. Robert Tavernor and Richard Schofield (Cambridge, Massachusetts: MIT Press, 1997).

[15] Cf. Kim Williams, "How Buildings Take Shape," Highlights for Children, 52, 3 (March 1997) 30-31.

[16] John Clagett, "Transformational Geometry and the Central European Church," Nexus: Architecture and Mathematics, Kim Williams, ed. (Fucecchio, Florence: Edizioni dell'Erba, 1996), 37-51.

[17] Alberto Pérez-Gómez, Architecture and the Crisis of Modern Science (Cambridge, Massachusetts: MIT Press, 1983) 18-39.

[18] Mario Salvadori, "Can there be any relationships between Mathematics and Architecture," Nexus: Architecture and Mathematics, Kim Williams, ed. (Fucecchio, Florence: Edizioni dell'Erba, 1996), 12.

Figure Captions

Figure 1. Interior of the Pantheon, Rome. Photograph courtesy of the Istituto Centrale per il Catalogo e la Documentazione, Rome.

Figure 2. The geometry of the interior of the Pantheon. Reproduced by permission of William MacDonald.

Figure 3. Piero della Francesca's geometrical method for constructing a square in perspective. The original square is defined by BCGF. The artist establishes three points: D, the spectator's distance from the square; A, the spectator's eye-level; A', the spectator's horizontal position with respect to the front edge of BCGF. AC intersect BF at point E, determining the rear limit of the square in perspective; lines A'B and A'C determine the left and right sides of the square in perspective and the length of back edge D'E'. Trapezoid BCE'D' represents the square BCGF in perspective. Reproduced by permission of Mark A. Peterson.

Figure 4. Above, axionometric section of Alberti's San Sebastiano with dimensions given in the local unit of measurement, the Mantuan *braccio*. Below, numerical progressions that include all dimensions, arranged with respect to individual elements of the church. Diagram reproduced by permission of Livio Volpi Ghirardini.

Figure 5. The volume of Michelangelo's Medici Chapel is determined by a double cube; its groundplan is a root-2 rectangle derived from one of the cubes. Figure by the author.

Figure 6. Successive divisions of the root-2 rectangle determines the musical intervals. This is the basis for determining significant dimensions of the Medici Chapel. Figure by the author.

Figure 7. Different configurations of truss construction are all based on triangulation. Reproduced by permission of *Highlights for Children.*